Paraproducts and Analysis of Rough Path

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Organizers:

Massimiliano Gubinelli
University of Bonn, Germany

Christoph Thiele
University of Bonn, Germany

[*]supported by Hausdorff Center for Mathematics, Bonn

Contents

1 Differential Equations Driven by Rough Paths: An Approach via Discrete Approximation

after A.M. Davie [1]
A summary written by Yujia Zhai

Abstract

We use Euler approximations to study systems of differential equations of the form $dy^i = \sum_j f^i_j(y)dx^i$, where $x(t)$ is assumed to have finite p-variation for some positive p. We will apply this approach to study the stochastic differential equations driven by Brownian motion.

1.1 Introduction to method of Euler approximations

One way to study the classical ordinary differential equations is to use Euler approximations associated with some partitions and show that these approximations converge to a limit which satisfies the equation as the partition gets finer. We will use this approach to study systems of differential equations of the form

$$dy^i = \sum_j^d f^i_j(y)dx^i, \qquad y^i(0) = y^i_0, \qquad i = 1, 2, \ldots, n \tag{1}$$

where $x(t)$ is a continuous vector-valued function of t which is only assumed to have finite p-variation, as opposed to be differentiable.

1.1.1 Notations

We say a continuous function $x(t) = (x^1(t), \ldots, x^d(t))$ defined on $[0, T]$ has finite p-variation if there is a continuous increasing function ω on $[0, T]$ such that $|x(t) - x(s)|^p \lesssim \omega(t) - \omega(s)$ for any $0 \le s \le t \le T$.

1.1.2 Discussion for $1 \le p < 2$

We will first clarify how to interpret (1) in the case when $1 \le p < 2$:

Definition 1. *We say $y(t)$ is a solution of (1) on $[0,T]$ if $y(0) = y_0$ and there exists a continuous increasing function $\tilde{\omega}$ on $[0,T]$ and a non-negative function θ on $[0,\infty)$ such that $\theta(\delta) = o(\delta)$ as $\delta \to 0$ and such that*

$$\left| y^i(t) - y^i(s) - \sum_{j=1}^{d} f^i_j(y(s))(x^j(t) - x^j(s)) \right| \leq \theta(\tilde{\omega}(t) - \tilde{\omega}(s)) \qquad (2)$$

The following theorem asserts the existence and uniqueness of the solution to (1):

Theorem 2. *Suppose $1 \leq p < \gamma \leq 2$.*
(i) Let $f \in C^{\gamma-1}$ and $y_0 \in \mathbb{R}^n$. Then there exists τ with $0 < \tau \leq T$, and a solution $y(t)$ of (1) for $0 \leq t < \tau$ such that if $\tau < T$ then $|y(t)| \to \infty$ as $t \to \tau$.
(ii) Let $f \in C^{\gamma}$ and $y_0 \in \mathbb{R}^n$. Then the solution of (1) given by (i) is unique in the sense that if $t < \tau$ in and $\tilde{y}$ is another solution of (1) on $[0,t]$, then $y = \tilde{y}$ on $[0,t]$.

We consider discrete approximations to a solution as follows: for $0 = t_0 < t_1 < \ldots < t_K = T$, let $x_k = x(t_k)$ and, given y_0, define y_k by the recurrence relation:

$$y^i_{k+1} = y^i_k + \sum_{j=1}^{d} f^i_j(y_k)(x^j_{k+1} - x^j_k), \quad i = 1, \ldots, n \qquad (3)$$

We shall need the following lemma about the approximations defined by (3):

Lemma 3. *(i) Suppose $f \in C^{\gamma-1}_0$. There exist positive constants C and M which depend only on $n, d, \gamma, p, \omega(t_K) - \omega(0)$ and $\|f\|_{\gamma-1}$, such that for any $0 \leq k \leq l \leq K$,*

$$\left| y^i_l - y^i_k - \sum_{j=1}^{d} f^i_j(y_k)(x^j_l - x^j_k) \right| \leq C(\omega(t_l) - \omega(t_k))^{\gamma/p}$$

and

$$|y_l - y_k| \leq M(\omega(t_l) - \omega(t_k))^{1/p}.$$

(ii) Suppose $f \in C^{\gamma}_0$. Let $\tilde{y_k}$ be the solution of (3) corresponding to the initial condition $\tilde{y_0} \in \mathbb{R}$. There is a constant $M > 0$ depending only on $n, d, \gamma, p, \omega(t_K) - \omega(0)$ and $\|f\|_{\gamma}$ such that for any $0 \leq k \leq K$,

$$|\tilde{y_k}^i - y^i_k| \leq M' max_i |\tilde{y_0}^i - y^i_0|.$$

We are now ready to construct the solution and prove Theorem 2.

Proof of Theorem. Consider $f_{(r)} \in C_0^{\gamma-1}$ such that $f_{(r)}(y) = f(y)$ for $|y| \leq r$. We take a sequence of partitions $(\mathcal{P}_m)_m$ of $[0, T]$ with mesh approaching 0, and let $\{y^{(m)}(s) : s \in \mathcal{P}_m\}$ be the solution of (3) using partition $\mathcal{P}_m$ (f is now replaced by $f_{(r)}$) . We then define $y(s)$ to be the limit(could be $\pm\infty$) of some subsequence of $(y^{(m)}(s))_m$ for $s \in \cup_m \mathcal{P}_m$. We shall note that approximations using (3) with $f_{(r)}$ and with f may be very different for y_l ($l > k$) given $|y_k| > r$, for some fixed partition. This indicates that the approximation defined by (3) with $f_{(r)}$ might not be efficient globally. To avoid this problem, we restrict our focus to $[0, t]$, where $t < \tau_r$ and $\tau_r := \sup\{0 \leq t < T :$ there exists m_0 such that $|y^{(m)}(s)| < r$ for all $m \geq m_0$ and $s \in \mathcal{P}_m$ with $0 \leq s \leq t\}$. τ_r is well-defined and positive for $r > |y_0|$ by Lemma 3 (ii). Also Lemma 3 (i) and passing to limits give $|y(s) - y(s')| \leq C(r, t)(\omega(s) - \omega(s'))^{1/p}$ for any $s, s' \in \mathcal{P}_m$ with $0 \leq s, s' \leq t < \tau_r$ and m large enough. Then by continuity, y can be extended to $[0, t]$. It follows from this construction that y satisfies (2) on $[0, t]$, for every $t < \tau_r$, and every r. Let $\tau = \lim_{r \to \infty} \tau_r$. Then y is a solution to (1) on $[0, \tau)$

It is noteworthy that while $y(t)$ is uniformly continuous on each $[0, \tau_r]$, it could blow up as $t \to \tau$. Indeed, it can be shown by contradiction that if $\tau < T$, this is exactly what happens, which we will not discuss here.

It is not hard to see that the following claim implies the uniqueness of the solution:

Claim: If $t < \tau$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that if $0 = t_0 < \ldots < t_K$ with $t_k - t_{k-1} < \delta$ for each k, then

$$|y_k - y(t_k)| < \epsilon$$

for each k, where y_k is given by (3).

Proof of Claim. There exists an r such that $|f| < r$ on $[0, t]$. Consider a partition $0 = t_0 < \ldots < t_K = T$. For $l \geq k$, let $z_l^{(k)}$ be the solution of (3) with $f_{(r)}$ and initial value $y(t_k)$. Then for any $0 \leq l \leq K$

$$|z_l^{(0)} - y(t_l)| \leq \sum_{k=1}^{l} |z_l^{(k-1)} - z_l^{(k)}| \lesssim \sum_{k=1}^{l} |z_k^{(k-1)} - y(t_k)| \lesssim \sum_{k=1}^{l} \theta(\tilde{\omega}(t_k) - \tilde{\omega}(t_{k-1}))$$

which tends to 0 as the partition gets finer. The second and third inequalities follow from Lemma 3 and (2). □

This completes the proof of Theorem 2. $\square$

1.1.3 Discussion for $2 < p \leq 3$

In this case, the approximation by (3) is not good enough, and we need a modified version:

$$y_{k+1}^i = y_k^i + \sum_{j=1}^d f_j^i(y_k)(x_{k+1}^j - x_k^j) + \sum_{j=1}^d \sum_{h=1}^n \partial_h f_j^i(y_k) \sum_{r=1}^d f_r^h(y_k) A^{rj}(t_k, t_{k+1}), \tag{4}$$

where $A^{rj}(s,t)$ is some quantity attempting to describe "$\int_s^t x^r(u) - x^j(s) dx^j(u)$". We require $A^{rj}(s,t)$ for $1 \leq r, j \leq d$ and $0 \leq s \leq t \leq T$ to satisfy:

(i) $A^{rj}(s,u) = A^{rj}(s,t) + A^{rj}(t,u) + (x^r(t) - x^r(s))(x^j(u) - x^j(t))$;

(ii) $|A^{rj}(s,t)|^{p/2} \leq \omega(t) - \omega(s)$.

Remark 4. *Intuitively (4) is a more accurate approximation because instead of approximating $f(y)$ by $f(y_k)$ for $y \in [t_k, t_{k+1}]$ as in (3),*

$$f_j^i(y) \approx f_j^i\left(y_k + \sum_{r=1}^d f_r(y_k)(x^r - x_k^r)\right) \approx f_j^i(y_k) + \sum_{h=1}^n \partial_h f_j^i(y_k) \sum_{r=1}^d f_r^h(y_k)(x^r - x_k^r)$$

It is therefore natural to re-interpret solution of (1) in the following way:

Definition 5. *We say $y(t)$ is a solution of (1) on $[0,T]$ if $y(0) = y_0$ and there exists a continuous increasing function $\tilde{\omega}$ on $[0,T]$ and a non-negative function θ on $[0,\infty)$ such that $\theta(\delta) = o(\delta)$ as $\delta \to 0$ and such that*

$$\begin{aligned} \Bigg| y^i(t) - y^i(s) - \sum_{j=1}^d f_j^i(y(s))(x^j(t) - x^j(s)) \\ - \sum_{j=1}^d \sum_{h=1}^n \partial_h f_j^i(y(s)) \sum_{r=1}^d f_r^h(y(s)) A^{rj}(s,t) \Bigg| \leq \theta(\tilde{\omega}(t) - \tilde{\omega}(s)) \end{aligned} \tag{5}$$

The following estimates verify the above intuition in the discrete case.

Suppose $2 < p < \gamma \leq 3$. For any partition $0 = t_0 < \ldots < t_K = T$ and solution of (4) corresponding to this partition, we have:

Lemma 6. *(i) Suppose* $f \in C_0^{\gamma-1}$. *There are positive constants* C, M, L *which depend only on* $n, d, \gamma, p, \omega(t_K) - \omega(0)$ *and* $\|f\|_{\gamma-1}$, *such that for any* $0 \le k \le l \le K$,

$$\left| y_l - y_k - \sum_{j=1}^{d} f_j^i(y_k)(x_l^j - x_k^j) \right.$$
$$\left. - \sum_{j=1}^{d} \sum_{h=1}^{n} \partial_h f_j^i(y_k) \sum_{r=1}^{d} f_r^h(y_k) A^{rj}(t_k, t_l) \right| \le M(\omega(t_l) - \omega(t_k))^{\gamma/p}$$

while

$$\left| y_l - y_k - \sum_{j=1}^{d} f_j^i(y_k)(x_l^j - x_k^j) \right| \le L(\omega(t_k) - \omega(t_l))^{2/p}$$

and

$$|y_l - y_k| \le C(\omega(t_l) - \omega(t_k))^{1/p}.$$

(ii) Suppose $f \in C_0^{\gamma}$. *Let* $\tilde{y_k}$ *be the solution of (3) corresponding to the initial condition* $\tilde{y_0} \in \mathbb{R}$. *There is a constant* $M > 0$ *depending only on* $n, d, \gamma, p, \omega(t_K) - \omega(0)$ *and* $\|f\|_\gamma$ *such that for any* $0 \le k \le K$,

$$|\tilde{y_k}^i - y_k^i| \le M' max_i |\tilde{y_0}^i - y_0^i|.$$

It is not hard to inspect that we have a similar result in the case $2 < p \le 3$ as in the case $1 < p \le 2$, which follows from Lemma 6 with the similar reasoning.

1.2 Application to SDE driven by Brownian motion

We can apply our method of Euler approximation to SDEs of the form: $dy^i = \sum_{j=1}^d f_j^i(y) dW^j$ with initial condition $y^i(0) = y_0^i$, for $i = 1, \ldots, n$, where (W^j) is a d-dimensional Brownian motion. Recall that with probability 1, W satisfies Hölder condition of exponent α for any $\alpha < \frac{1}{2}$, i.e. W has finite p-variation for $p > 2$ with $\omega(t) = t$. We will focus on the following theorem, which indicates how to use Euler approximation (discussed in section (1.1.3)) in SDE theory. We first clarify that from now on, $A^{rj}(s,t) := \int_s^t W^r(u) - W^r(s) dW^j(u)$, where in the integral is a standard Itô integral.

Theorem 7. *Suppose* $f \in C^\gamma$ *where* $\gamma > 1$. *Then, with probability 1, for any choice of* y_0 *the Itô equation* $dy^i = \sum_{j=1}^d f_j^i(y) dW^j$ *with* $y(0) = y_0$ *either* *has a solution in the sense of Definition 5 for all $t \geq 0$ or, for some $T > 0$, a solution on $0 \leq t < T$ with $|y(t)| \to \infty$ as $t \to T$. Moreover the solution is unique in the sense that if $\tilde{y}$ is another solution on $[0, \tau)$ in the sense of Definition 5, then $y = \tilde{y}$ on $[0, \tau)$.*

Proof. We take for granted the fact that with probability 1, the Itô equation assuming $f \in C_0^\gamma$ has a continuous solution flow $(s, t, x) \to F(s, t, x) \in \mathbb{R}^d$, defined for $s < t$ and $x \in \mathbb{R}^d$ such that for any choice of s, t, x the solution of the equation with $y(s) = x$ satisfies $y(t) = F(s, t, x)$ with probability 1. Here $y(t)$ represents for the standard solution of the Itô SDE. One important property for F is that it is a locally Lipschitz function of x, with uniform Lipschitz bounds on compact sets. The following lemma gives some estimates about F, and will be used to prove the theorem:

Lemma 8. *Fix $T > 0$ and $1 < q < \alpha = (1 + \gamma)/2$. Let*

$$Z^i(s, t, y) := F^i(s, t, y) - y^i - \sum_{j=1}^{d} f_j^i(y)(W^j(t) - W^j(s)) - \sum_{j=1}^{d} \sum_{h=1}^{n} \partial_h f_j^i(y) \sum_{r=1}^{d} f_r^h(y) A^{rj}(s, t).$$

Then for any $0 \leq s < t \leq T$,

(i) $\mathbb{E} \left| Z^i(s, t, y) \right|^k \leq C(T, f)(t - s)^{k\alpha}$ *for any $k \geq 2$;*

(ii) If we further assume $|y| < L$, for some L, then with probability 1, there is a constant C' depending on $T, L, f, w \in \Omega$ such that $|Z^i(s, t, y)| \leq C'(t - s)^q$.

Remark 9. *We will omit the proof of the lemma, but it is notable that good estimate on average for $|Z(s, t, y)|$ in (i) foreshadows some good result for $|Z(s, t, y)|$ on a large portion of the probability space, which is stated in (ii). Here, "good" refers to the exponent q being greater than 1, which will play an important role in the proof.*

Since the existence of the solution flow and the estimates in the lemma require f to be compactly supported, we thus consider $f^{(m)} \in C_0^\gamma$ such that $f^{(m)}(y) = f(y)$ for $|y| \leq m$, and its associated flow $F^{(m)}(s, t, y)$ for $m = 1, 2, \ldots$. Then with probability 1, there is a sequence (C_m) such that

$$\left| Z^{(m)}(s, t, y) \right| \leq C_m (t - s)^q \quad \text{and} \tag{6}$$

$$\left|F^{(m)}(s,t,x) - F^{(m)}(s,t,y)\right| \leq C_m|x-y| \tag{7}$$

for $m \in \mathbb{N}, 0 \leq s < t < m$ and $|x|, |y| < m$, where (6) and (7) follow from Lemma 8 and Lipschitz property of $F^{(m)}$, respectively.

We can now fix a Brownian path such that the above conditions hold. The existence of the solution corresponding to this particular path is a consequence of the discussion in section (1.1.3). To prove the uniqueness of the solution with respect to this path, we assume that $\tilde{y}$ is another solution on $[0,\tau)$ which is not identical to y on $[0,\tau)$ (either $\tau \leq \infty$ or $\tau \leq T$). Let $\tau_1 := \sup\{t \geq 0 : y(s) = \tilde{y}(s) \text{ for } 0 \leq s < t\}$, and $y_0 := y(\tau_1) = \tilde{y}(\tau_1)$. Now we fix any τ' with $\tau_1 < \tau' < \tau$ and choose m such that $|y| < m$ and $|\tilde{y}| < m$ on $[\tau_1, \tau']$.

Claim : $y(t) = F^{(m)}(\tau_1, t, y_1)$ for $\tau_1 \leq t \leq \tau'$.

Proof of Claim. Fix $t \in [\tau_1, \tau']$, and pick $N \in \mathbb{N}$. Consider $\tau_1 = t_0 < t_1 < \ldots < t_N = t$ such that $t_k - t_{k-1} < N^{-1}$. Now by (5)

$$\Bigg| y^i(t_{k+1}) - y^i(t_k) - \sum_{j=1}^{d} f_j^{(m)i}(y(t_k))(W^j(t_{k+1}) - W^j(t_k))$$
$$- \sum_{j=1}^{d}\sum_{h=1}^{n} \partial_h f_j^{(m)i}(y(t_k)) \sum_{r=1}^{d} f_r^{(m)h}(y(t_k)) A^{rj}(t_k, t_{k+1}) \Bigg| \leq \theta(\tilde{\omega}(t_{k+1}) - \tilde{\omega}(t_k))$$

This, along with (6), gives

$$\left|y(t_{k+1}) - F^{(m)}(t_k, t, y(t_k))\right| \leq \theta(\tilde{\omega}(t_{k+1}) - \tilde{\omega}(t_k)) + C_m N^{-q}$$

Then

$$\begin{aligned}|y(t) - F^{(m)}(\tau_1, t, y_0)| &\leq \sum_{k=0}^{N-1} \left|F^{(m)}\left(t_{k+1}, t, y(t_{k+1})\right) - F^{(m)}\left(t_{k+1}, t, F^{(m)}(t_k, t_{k+1}, y(t_k)\right)\right| \\ &\leq C_m \left(\sum_{k=0}^{N-1} \theta(\tilde{\omega}(t_{k+1}) - \tilde{\omega}(t_k)) + C_m N^{1-q}\right),\end{aligned}$$

which goes to 0 as $N \to \infty$ (which shows the importance of $q > 1$). The last inequality follows from (7). □

We can apply the same reasoning for $\tilde{y}$. The the deterministic property of the solution flow with a fixed path gives a contradiction to the definition of τ_1. □

References

[1] Davie A.M., *Differential equations driven by rough paths: an approach via discrete approximation.* 2000 Mathematics Subject Classification. Primary 60H10. Secondary 34F05.

Yujia Zhai, Cornell University
email: **yz733@cornell.edu**

2 Controlling Rough Paths. Part I

after M. Gubinelli [1]
A summary written by Cristina Benea

Abstract

By using algebraic tools, M. Gubinelli [1] gives meaning to indefinite integration with respect to an irregular function, under certain analytic constraints. We sketch the main ideas behind this novel approach, that allows to recover Young's theory of integration, as well as the main results of Lyons' theory of rough paths in Hölder topology.

2.1 Introduction

The theory of *rough paths* was initiated by the study of differential equations "driven by rough signals" (see [2]), such as

$$\begin{cases} dY_t^\mu = \sum_\mu \varphi(Y_t)_\nu^\mu dX_t^\nu \\ Y_{t_0} = y, \end{cases} \tag{1}$$

where $X : J \to V$ is γ-Hölder continuous, and $\varphi \in C(V, V \otimes V^*)$. Here $J \subseteq \mathbb{R}$ is a fixed interval, and V a vector space. By definition, a solution Y is a γ-Hölder continuous map $Y \in \mathscr{C}^\gamma(J, V)$ such that

$$Y_t^\mu = y + \int_{t_0}^t \varphi(Y_u)_\nu^\mu dX_u^\mu, \qquad \forall t \in J. \tag{2}$$

Before applying any standard differential equations techniques for proving existence or uniqueness (this will be explained in Part II), we will need to make sense of the integral

$$\int_s^t F_u dX_u, \tag{3}$$

with minimal assumptions on X.

In [2], Lyons develops a way of understanding such expressions for paths X of finite p-variation, for some $p < \infty$. The iterated integrals of X play an important role in this theory, and in some sense they help characterize the rough path X. The take is that, in order to define the integral above for a non-smooth path X, other quantities related to X should be involved. In [1], Gubinelli uses a "tensor process" $\mathbb{X}^2 : J \times J \to V \otimes V$ for defining the integral $\int_s^t Z_u dW_u$ for paths Z and W *weakly controlled by* X (that is, for paths whose increments are controlled by X). Here the roughness is measured by Hölder-like (semi)norms. The techniques used for defining integration over rough paths seem to originate from homological algebra.

Before presenting the main results and the means to obtain them, we recall a few particular situations in which the integral $\int_s^t F_u dX_u$ from (3) is long understood. If F is continuous and X has bounded variation, it represents the classical Riemann-Stieltjes integral. If F is γ_1-Hölder continuous and X is γ_2-Hölder continuous, with $\gamma_1 + \gamma_2 > 1$, the integral (3) was defined in [3] by Young. The interest for paths that are less regular is motivated by stochastic differential equations, where often the "driving signal" X is a Brownian rough path or a similar process.

2.2 Algebraic Prelude

The idea is to represent the integral $\int_{t_0}^t F_u dX_u$ as a (possibly unique) solution of a certain algebraic equation. The algebraic spaces will play an important role and they will contain certain analytic information.

The most basic example is that of the integral $\int_{t_0}^t f(u)dx(u)$, when f is a bounded continuous function on $\mathbb{R}$ and x is a function on $\mathbb{R}$ with continuous first derivative. Then $A(t) = \int_{t_0}^t f(u)dx(u)$, together with $R(s,t) = \int_s^t (f(u) - f(s))\, dx(u)$ represent the unique solution $(A, R) \in C^1(\mathbb{R}) \times C(\mathbb{R}^2)$ such that

$$f(s)\,(x(t) - x(s)) = A(t) - A(s) - R(s,t), \tag{4}$$

with $A(t_0) = 0$ and $\lim\limits_{t \to s} \frac{R(s,t)}{|t-s|} = 0$.

Now we describe the abstract setting from [1]: $\mathscr{C}$ denotes the algebra of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}$ (later on, $\mathscr{C}(J,V)$ will denote the algebra of bounded continuous functions from the interval $J \subseteq \mathbb{R}$ to the Banach space V) and

$$\Omega\mathscr{C}_n := \{R : \mathbb{R}^{n+1} \to \mathbb{R} \text{ bounded continuous functions }, R_{t,\ldots,t} = 0\}.$$

Elements of $\Omega\mathscr{C}_n$ are called *processes*, and are to be distinguished from *paths*, which are the elements of $\mathscr{C}$. The subspace of γ-Hölder continuous paths is denoted $\mathscr{C}^\gamma$, and $\Omega\mathscr{C}^\gamma$ is the subspace of processes $X \in \Omega\mathscr{C}$ so that

$$\|X\|_\gamma := \sup_{s,t\in\mathbb{R}} \frac{|X_{st}|}{|t-s|^\gamma} < \infty.$$

Finally, $\Omega\mathscr{C}_2^\gamma$ denotes the subspace of processes in $\Omega\mathscr{C}_2$ which are finite linear combinations of elements $X \in \Omega\mathscr{C}_2$ with the property that

$$\|X\|_{\gamma_1,\gamma_2} = \sup_{s,u,t\in\mathbb{R}} \frac{|X_{sut}|}{|u-s|^{\gamma_1}|t-u|^{\gamma_2}} < \infty,$$

for some $0 < \gamma_1, \gamma_2 < \gamma$ with $\gamma_1 + \gamma_2 = \gamma$.

With the definitions above, we are ready to bring forward the algebraic machinery. We introduce two linear maps, $\delta : \mathscr{C} \to \Omega\mathscr{C}$ and $N : \Omega\mathscr{C} \to \Omega\mathscr{C}_2$, which are defined by

$$(\delta A)_{st} := A_t - A_s, \qquad (NR)_{sut} := R_{st} - R_{ut} - R_{su}.$$

These are just the first two of the so-called "coboundary" operators $\partial^* : \Omega\mathscr{C}_n \to \Omega\mathscr{C}_{n+1}$ associated to a cochain complex

$$(\partial^* A)_{t_1\ldots t_{n+1}} := \sum_{i=1}^{n+1} (-1)^i A_{t_1\ldots\hat{t_i}\ldots t_{n+1}},$$

and which have the property that $\partial^*\partial^* = 0$ (or equivalently, $\operatorname{Im}\partial^*|_{\Omega\mathscr{C}_n} \subseteq \operatorname{Ker}\partial^*|_{\Omega\mathscr{C}_{n+1}}$). In fact, it turns out that $\operatorname{Im}\partial^*|_{\Omega\mathscr{C}_n} = \operatorname{Ker}\partial^*|_{\Omega\mathscr{C}_{n+1}}$, or in other words, the cochain complex

$$0 \to \mathbb{R} \xrightarrow{\partial^*} \Omega\mathscr{C} \xrightarrow{\partial^*} \Omega\mathscr{C}_2 \xrightarrow{\partial^*} \ldots$$

is exact at every $\Omega\mathscr{C}_n$. The advantage is that deciding if $X \in \operatorname{Im}\partial^*|_{\Omega\mathscr{C}_n}$ (which is an existence result) becomes equivalent to checking whether $\partial^*|_{\Omega\mathscr{C}_{n+1}}(X) = 0$, and the first is more difficult than the latter.

Equation (4) can be reinterpreted as $f_s(\delta x)_{st} = (\delta A)_{st} - R_{st}$, and the problem of defining the integral can be reformulated in the following way:

Question 1. *Given two paths $F, X \in \mathscr{C}$, can one find a (possibly) unique decomposition $F\delta X = \delta A - R$, with $A \in \mathscr{C}$ and $R \in \Omega\mathscr{C}$?*

Even though this is not achievable without imposing supplementary conditions on F and X, we note that, once we have R as above, the existence of A is equivalent to $F\delta X + R$ belonging to $\mathrm{Im}\ \delta = \mathrm{Ker}\ N$. So in fact we have to check that

$$N\left(F\delta X + R\right) = 0.$$

This is a consequence of the fact that

$$0 \to \mathscr{C}/\mathbb{R} \xrightarrow{\delta} \Omega\mathscr{C} \xrightarrow{N} N\left(\Omega\mathscr{C}\right) \to 0$$

is a short exact sequence. A decomposition as in Question 1 is equivalent to an isomorphism $\Omega\mathscr{C} \simeq \delta\mathscr{C} \oplus \mathscr{R}$. For an exact sequence as above, this corresponds to a *right split*, i.e. the existence of a map $\Lambda : N\left(\Omega\mathscr{C}\right) \to \Omega\mathscr{C}$ so that $N\Lambda = id_{N(\Omega\mathscr{C})}$.

As mentioned before, this is not possible without additional assumptions on F and X; instead, we will be looking for a suitable linear subspace $\mathscr{E} \subseteq \Omega\mathscr{C}$ containing $\delta\mathscr{C}$, so that the short exact sequence

$$0 \to \mathscr{C}/\mathbb{R} \xrightarrow{\delta} \mathscr{E} \xrightarrow{N} N\mathscr{E} \to 0$$

splits to the right (i.e. there exists a linear map $\Lambda_{\mathscr{E}} : N\mathscr{E} \to \mathscr{E}$ so that $N\Lambda_{\mathscr{E}} = id_{N\mathscr{E}}$), and as a consequence, $\mathscr{E} \simeq \delta\mathscr{C} \oplus N\mathscr{E}$. Then we can represent δA as

$$\delta A = F\delta X - \Lambda_{\mathscr{E}} N(F\delta X), \tag{5}$$

provided $F\delta X \in \mathscr{E}$. A good premise is that $N\mathscr{E}$ should be of the form $\Omega\mathscr{C}^z$, for some $z > 1$, since then $\mathrm{Im}\,\delta \cap \Omega\mathscr{C}^z = \{0\}$.

Proposition 1. *If $z > 1$, then there exists a unique linear map $\Lambda : N(\Omega\mathscr{C}) \cap \Omega\mathscr{C}_2^z \to \Omega\mathscr{C}^z$ such that $N\Lambda = id_{N(\Omega\mathscr{C})\cap\Omega\mathscr{C}_2^z}$ and, for all $W \in N(\Omega\mathscr{C}) \cap \Omega\mathscr{C}_2^z$ we have*

$$\|\Lambda W\|_z \leq \frac{1}{2^z - 2} \sum_{i=1}^{n} \|W_i\|_{\rho_i, z-\rho_i},$$

if $W = \sum_{i=1}^n W_i$, with $\|W_i\|_{\rho_i,z-\rho_i} < \infty$.

2.3 Young's Theory of Integration

The above machinery, and especially Proposition 1 now allow to give a straightforward proof of the existence of the Young integral from [3]:

Proposition 2. *Fix an interval $I \subseteq \mathbb{R}$. If $F \in \mathscr{C}^{\gamma_1}(I)$ and $X \in \mathscr{C}^{\gamma_2}(I)$ with $\gamma = \gamma_1 + \gamma_2 > 1$, define*

$$\int_s^t F_u dX_u := [F\delta X - \Lambda N(F\delta X)]_{st}, \qquad s,t \in I,$$

where $\Lambda : N(\Omega\mathscr{C}) \cap \Omega\mathscr{C}_2^\gamma \to \Omega\mathscr{C}^\gamma$ is given by Proposition 1. Moreover, we have

$$\left| \int_s^t (F_u - F_s) dX_u \right| \le \frac{1}{2^\gamma - 2} |t-s|^\gamma \|F\|_{\gamma_1,I} \|G\|_{\gamma_2,I}, \qquad \forall s,t \in I.$$

The equivalence of the above definition with Young's theory of integration is established by the following

Corollary 3.

$$\int_s^t F_u dX_u = \lim_{|\Pi| \to 0} \sum_{\{t_i\} \in \Pi} F_{t_i}(X_{t_{i+1}} - X_{t_i}), \qquad s,t \in I,$$

where the limit is taken over partitions $\Pi = \{t_0, \ldots, t_n\}$ of the interval $[s,t] \subseteq I$ such that $t_0 = s, t_n = t, t_{i+1} > t_i, |\Pi| = \sup_i |t_{i+1} - t_i|$.

2.4 More irregular paths

This is the most technical section of the first part, in which the integral $\int_s^t Z_u dW_u$ is defined for rough paths Z and W whose increments are controlled by a specific rough path X. It is not clear how to generalize the theory beyond this point.

The framework is also more general; $X \in \mathscr{C}^\gamma(V)$ is a γ-Hölder continuous path in a Banach space V (considered to be finite dimensional so that algebraic and analytical objects are both well defined), and $\mathbb{X}^2 \in \Omega\mathscr{C}^{2\gamma}(V \otimes V)$ is a tensor process so that

$$N(\mathbb{X}^{2,\mu\nu})_{sut} = (\delta X^\mu)_{su}(\delta X^\nu)_{ut}. \tag{6}$$

First, the integral $\int X dX$ is defined (uniquely, only if $\gamma > \frac{1}{2}$). This extends easily to $\int Y dX$ when $Y_t^\mu = A_\nu^\mu X_t^\nu$, where A is a linear application from V to V. Eventually, $\int Y dX$ is defined for paths Y which are only *locally* linear functions of X, i.e.

$$\delta Y^\mu = G_\nu^\mu \delta X^\nu + Q^\mu,$$

where Q is a remainder in $\Omega\mathscr{C}(V)$ and G is a path in $\mathscr{C}(V \otimes V^*)$. Here we use the Einstein summation convention. All of the above direct to the definition

Definition 4. *Let $I \subseteq \mathbb{R}$ and $X \in \mathscr{C}^\gamma(I, V)$. A path $Z \in \mathscr{C}^\gamma(I, V)$ is said to be* weakly controlled by X in I with remainder of order η *if there exists a path $Z' \in \mathscr{C}^{\eta-\gamma}(I, V \otimes V^*)$ and a process $R_Z \in \Omega\mathscr{C}^\eta(I, V)$ with $\eta > \gamma$ such that $\delta Z^\mu = Z'^{\mu\nu} \delta X^\nu + R^\mu_Z$. In this case, we write $(Z, Z') \in \mathscr{D}^{\gamma,\eta}_X(I, V)$.*

Theorem 5. *For every $(Z, Z') \in \mathscr{D}^{\gamma,\eta}_X(I, V)$ and $(W, W') \in D^{\gamma,\eta}_X(I, V)$ with $\eta + \gamma = \sigma > 1$, define for all $s, t \in I$*

$$\int_s^t Z^\mu_u dW^\nu_u := Z^\mu_s \delta W^\nu_{st} + Z'^\mu_{\mu',s} W'^\nu_{\nu',s} \mathbb{X}^{2,\mu'\nu'}_{st} - \Big[\Lambda N \left(Z^\mu \delta W^\nu + Z'^\mu_{\mu'} W'^\nu_{\nu'} \mathbb{X}^{2,\mu'\nu'}_{\nu'} \right)\Big]_{st}.$$

Moreover, the bilinear application from $D^{\gamma,\eta}_X(I, V) \times D^{\gamma,\eta}_X(I, V)$ to $D^{\gamma,\min(2\gamma,\eta)}_X(I, V \otimes V)$ given by

$$((Z, Z'), (W, W')) \mapsto \left(\int_0^\cdot Z dW, Z'W' \right)$$

is continuous. This definition coincides with the classical integral of Young if Z and W are regular enough.

Application to differential equations driven by rough paths are discussed in the following section, which is contained in Part II of [1]. Connections to Terry Lyons' theory of rough paths from [2] are discussed in the last section of Part II.

References

[1] Gubinelli, M., *Controlling Rough Paths.* Journal of Functional Analysis 216, 86–140 (2004).

[2] Lyons, T., *Differential Equations Driven by Rough Signals* Rev. Mat. Iberoam. 14 no. 2, 215 –310 (1998).

[3] Young, L.C., *An Inequality of the Hölder Type, Connected with Stieltjes Integration* Acta Math. 67 (1936).

Cristina Benea, Université de Nantes
email: **cristina.benea@univ-nantes.fr**

3 Controlling rough paths (part II)

after M. Gubinelli [1]
A summary written by Gianmarco Brocchi

Abstract

We study the problem of existence, uniqueness and continuity of solution of differential equations driven by irregular paths with Hölder exponent greater than $\frac{1}{3}$ (e.g. samples of Brownian motion). We will also show connection with known stochastic integrals.

3.1 Introduction

Consider an interval $J \subseteq \mathbb{R}$ and a γ-Hölder path X in $\mathcal{C}^\gamma(J, V)$ taking values in a finite dimensional vector space V. Let φ be a function in $C(V, V \otimes V^\star)$. We are interested in studying the controlled differential equation

$$\boxed{dY_t^\mu = \varphi(Y_t)^\mu_\nu \, dX_t^\nu} \quad Y_{t_0} = y, \ t_0 \in J \tag{1}$$

where μ, ν are vector indices[1]. A solution to (1) will be a continuous path $Y \in \mathcal{C}^\gamma(J, V)$ such that

$$Y_t^\mu = y + \int_{t_0}^t \varphi(Y_u)^\mu_\nu \, dX_u^\nu \tag{2}$$

for every $t \in J$. When $\gamma > \frac{1}{2}$ sufficient conditions on φ allow to consider integral in (2) as a Young integral. When $\frac{1}{2} \geq \gamma > \frac{1}{3}$ the integral must be understood as integral of a weakly-controlled path, as in part I of [1].
In this last case, given a rough path $(X, \mathbb{X}^2)$, the solution of the differential equation (1) driven by X is a weakly-controlled path in $\mathcal{D}_X^{\gamma,2\gamma}(J, V)$.

To prove these results we will show that the solution map

$$Y \mapsto G(Y)_t = Y_{t_0} + \int_{t_0}^t \varphi(Y_u)^\mu_\nu \, dX_u^\nu \tag{3}$$

is locally a strict contraction on a subset of the Banach space $\mathcal{C}^\gamma(J, V)$ of Hölder continuous functions on J to a finite vector space V. Therefore it has an unique fixed-point. Moreover, the Itô map $Y = F(y, \varphi, X)$ which sends

[1] We will use Einstein notation omitting summation over repeated indices.

the data of the differential equation to the solution is Lipschitz continuous (on compact intervals J) in each argument.

The following table, where $\delta \in (0,1)$, sums up sufficient hypothesis and our main results in the two cases:

	$\gamma > 1/2$	$1/2 \geq \gamma > 1/3$
Integral in (2)	Young integral	Integral based on $(X, \mathbb{X}^2)$
Solution	$Y \in \mathcal{C}^\gamma$	$Y \in \mathcal{D}_X^{\gamma,2\gamma}$
Conditions for Existence	$\varphi \in C^\delta(V, V \otimes V^\star)$, $(1+\delta)\gamma > 1$	$\varphi \in C^\delta(V, V)$, $(2+\delta)\gamma > 1$
Stronger condition for Uniqueness	$\varphi \in C^{1,\delta}(V, V \otimes V^\star)$	$\varphi \in C^{2,\delta}(V, V)$

3.2 Preliminaries

We indicate a bounded function on $\mathbb{R}^2$ as element in $\Omega\mathcal{C}$. For a such function we can consider the norm:

$$\|A\|_\gamma := \sup_{s,t \in \mathbb{R}^2} \frac{|A_{st}|}{|t-s|^\gamma}.$$

The space $\Omega\mathcal{C}^\gamma$ is the subspace of $\Omega\mathcal{C}$ such that $\|A\|_\gamma < \infty$.
For a path X on $I \subset \mathbb{R}$, the map $(\delta X)_{st} := X_t - X_s$ maps $\mathcal{C}^\gamma$ to $\Omega\mathcal{C}^\gamma$.

Lemma 1. *Let $I = [a,b]$ and $\gamma, \eta \in \mathbb{R}$. If $\gamma < \eta$ then*

$$\|\cdot\|_{\gamma,I} \leq |b-a|^{\eta-\gamma} \|\cdot\|_{\eta,I}$$

i.e. the inclusion $\Omega\mathcal{C}^\eta(I) \hookrightarrow \Omega\mathcal{C}^\gamma(I)$ is continuous.

Lemma 2. *Let I, J two adjacent intervals on $\mathbb{R}$ and let X be a path in $\mathcal{C}^\gamma(I,V)$ and in $\mathcal{C}^\gamma(J,V)$. If $NX \in \mathcal{C}^{\gamma_1,\gamma_2}(I \cup J, V)$, with $\gamma_1 + \gamma_2 = \gamma$, then*

$$\|X\|_{\gamma, I\cup J} \leq 2(\|X\|_{\gamma,I} + \|X\|_{\gamma,J}) + \|NX\|_{\gamma_1,\gamma_2,I\cup J}$$

and $X \in \mathcal{C}^\gamma(I \cup J, V)$.

3.3 Existence and uniqueness when $\gamma > \frac{1}{2}$

Proposition 3 (Existence). *If $\gamma > 1/2$ and $\varphi \in C^{\delta}(V, V \otimes V^{\star})$, $\delta \in (0,1)$ with $(1+\delta)\gamma > 1$, there exists a path $Y \in \mathcal{C}^{\gamma}(J, V)$ solution of the differential equation (1). (The integral in (2) must be understood as Young integral.)*

Sketch of the proof. Start with an interval $I = [t_0, t_0 + T] \subseteq J$ for $T > 0$. Under the condition $(1+\delta)\gamma > 1$, G maps $\mathcal{C}^{\gamma}(I, V)$ to itself. Using decomposition of path in $\mathcal{C}^{\gamma}$ we can fix a compact, convex subset Q_I which is invariant under G. The map G is proved to be continuous on Q_I so, by Leray-Schauder-Tychnoff theorem, exists a fixed-point for G in Q_I. We conclude covering J with interval I and patching together local solutions using Lemma 2. □

Proposition 4 (Uniqueness). *If $\gamma > 1/2$, $\varphi \in C^{1,\delta}(V, V \otimes V^{\star})$, $\delta \in (0,1)$ with $(1+\delta)\gamma > 1$, there exists a* unique *solution Y in $\mathcal{C}^{\gamma}(J, V)$ of the differential equation (1). The Itô map $F(y, \varphi, X)$ is Lipschitz continuous in the following sense:*

$$\|F(y, \varphi, X) - F(\widetilde{y}, \widetilde{\varphi}, \widetilde{X})\|_{\gamma,J} \leq M(\|X - \widetilde{X}\|_{\gamma,J} + \|\varphi - \widetilde{\varphi}\|_{1,\delta} + |y - \widetilde{y}|)$$

with a constant M depending only on $\|X\|_{\gamma,J}$, $\|\widetilde{X}\|_{\gamma,J}$, $\|\varphi\|_{1,\delta}$, $\|\widetilde{\varphi}\|_{1,\delta}$ and J.

Idea of the proof. For $T < 1$ we can fix an invariant compact set Q_I as in the previous Proposition. For T small enough G is proved to be a locally strict contraction on Q_I, this means we can take $\alpha = \alpha(T) < 1$ such that

$$\|G(Y) - G(\widetilde{Y})\|_{\gamma,I} \leq \alpha \|Y - \widetilde{Y}\|_{\gamma,I}$$

when $Y, \widetilde{Y} \in Q_I$ and $X = \widetilde{X}$. Then G is a strict contraction on Q_I and it has a unique fixed-point. Again, we extend the unique solution to $J \supset I$. □

3.4 Existence and uniqueness when $\frac{1}{2} \geq \gamma > \frac{1}{3}$

Proposition 5 (Existence). *If $\gamma > 1/3$, $\varphi \in C^{1,\delta}(V, V)$, $\delta \in (0,1)$ with $(2+\delta)\gamma > 1$, there exists a weakly-controlled path Y in $\mathcal{D}_X^{\gamma,2\gamma}(J, V)$ solution of the differential equation (1). (The integral in (2) must be understood as based on the couple $(X, \mathbb{X}^2)$.)*

Sketch of the proof. The path $\varphi(Y)$ lives in $\mathcal{D}_X^{\gamma,(1+\delta)\gamma}(J, V)$,. Integration against X makes sense for $(2+\delta)\gamma > 1$. We claim, similarly to Prop 3, that G maps $\mathcal{D}_X^{\gamma,2\gamma}(I, V)$ to itself. Using decomposition for $Z = G(Y)$:

$$\delta Z^{\mu} = Z'_{\nu} \delta X^{\nu} + R_Z^{\mu} = \varphi(Y)^{\mu}_{\nu} \delta X^{\nu} + \partial^{\kappa} \varphi(Y)^{\mu}_{\nu} Y'^{\kappa}_{\rho} \mathbb{X}^{2,\nu\rho} + Q_Z^{\mu}$$

we bound the norm $\|Z\|_{*,I} = \|Z\|_{\mathcal{D}_X(\gamma,2\gamma,I)}$. In the bound we get we can choose a time $T_* < 1$ such that for all $T < T_*$ we have a set Q'_I invariant under G.
Then exists a solution in $\mathcal{D}_X^{\gamma,2\gamma}(I,V)$ for any $I \subseteq J$ small enough. Consider a covering of J with $I_1, \dots, I_n$ suitable intervals. Patching together local solutions we get a global one $\overline{Y}$ defined on $\cup_i I_i = J$. Again, use Lemma 2 iteratively to prove that $\overline{Y}$ belongs to $\mathcal{D}_X^{\gamma,2\gamma}(J,V)$. □

Proposition 6 (Uniqueness)**.** *If* $\gamma > 1/3$, $\varphi \in C^{2,\delta}(V,V)$, $\delta \in (0,1)$ *with* $(2+\delta)\gamma > 1$*, there exists a* unique *solution* $Y \in \mathcal{D}_X^{\gamma,2\gamma}(J,V)$ *of the differential equation* (1)*, where the integral in* (2) *is based on the couple* $(X, \mathbb{X}^2)$*. Moreover, the Itô map* $F(y,\varphi,X,\mathbb{X}^2)$ *is Lipschitz continuous.*

Idea of the proof. As in Proposition 4 we use decompositions in order to bound

$$\epsilon_{Z,I} = \|\varphi(Y) - \widetilde{\varphi}(\widetilde{Y})\|_{\infty,I} + \|\varphi(Y) - \widetilde{\varphi}(\widetilde{Y})\|_{\gamma,I} + \|R_Z - R_{\widetilde{Z}}\|_{2\gamma,I}.$$

We get an invariant set under G. Choosing $T < 1$ small enough we obtain a $\alpha < 1$ such that

$$\|G(Y) - G(\widetilde{Y})\|_{*,I} \le \alpha \, \|Y - \widetilde{Y}\|_{*,I}.$$

Then G is a strict contraction in $\mathcal{D}_X^{\gamma,2\gamma}(I,V)$ and has a unique fixed-point. Patching together local solutions we get a global one defined on J in $\mathcal{D}_X^{\gamma,2\gamma}(J,V)$. □

3.5 Some probability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and let X be a standard Brownian motion defined on it taking values in $V = \mathbb{R}^n$. For a fixed $\gamma < \frac{1}{2}$ and a bounded interval I, the process X is almost surely locally γ-Hölder continuous, then we can choose a version of X in $\mathcal{C}^\gamma(I,V)$, .

Via stochastic integration we can define

$$W^{\mu\nu}_{\text{Itô},st} := \int_s^t (X^\mu_u - X^\mu_s)\, \hat{d}X^\nu_u$$

in the sense of the Itô integral (indicated by the hat in $\hat{d}X^\nu_u$) with respect to the filtration $\mathcal{F}_t = \sigma(X_s\,;\, s \le t)$. For any $s,u,t \in \mathbb{R}$ we have:

$$W^{\mu\nu}_{\text{Itô},st} - W^{\mu\nu}_{\text{Itô},su} - W^{\mu\nu}_{\text{Itô},ut} = (X^\mu_u - X^\mu_s)(X^\nu_t - X^\nu_u) \tag{4}$$

then we can consider a continuous version $\mathbb{X}^2_{\text{Itô}}$ of the process $(s,t) \mapsto W_{\text{Itô},st}$ such that (4) holds almost surely for all $s,u,t \in \mathbb{R}$.

Using a variation of an argument introduced in [2],[3] to control Hölder-like seminorms of continuous stochastic processes with a corresponding integral norm, it is possible to show that $\mathbb{X}^2_{\text{Itô}}$ belongs to $\Omega\mathcal{C}^{2\gamma}(I, V \otimes V)$.

We introduce also the Stratonovich integral:

$$\mathbb{X}^{2,\mu\nu}_{\text{Strat.},st} := \int_s^t (X^\mu_u - X^\mu_s) \circ \hat{d}X^\nu_u.$$

From stochastic integration we know that

$$\mathbb{X}^{2,\mu\nu}_{\text{Strat.},st} = \mathbb{X}^{2,\mu\nu}_{\text{Itô},st} + \frac{g^{\mu\nu}}{2}(t-s)\,, \quad \text{where } g^{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Also in this case we can select a continuous version of $\mathbb{X}^2_{\text{Strat.},st}$ in $\Omega\mathcal{C}^{2\gamma}$ such that (4) holds for it.

We have introduced the following integrals so far:

Itô integral $\delta I^\mu_{\text{Itô},st} = \int_s^t \varphi(X_u)^\mu_\nu \, \hat{d}X^\nu_u$	Integral based on $(X, \mathbb{X}^2_{\text{Itô}})$ $\delta I^\mu_{\text{rough},st} = \int_s^t \varphi(X_u)^\mu_\nu \, dX^\nu_u$
Stratonovich integral $\delta I^\mu_{\text{Strat.},st} = \int_s^t \varphi(X_u)^\mu_\nu \circ \hat{d}X^\nu_u$	Integral based on $(X, \mathbb{X}^2_{\text{Strat.}})$ $\delta J^\mu_{st} = \int_s^t \varphi(X_u)^\mu_\nu \, dX^\nu_u$

The connection between them is pointed out by following theorem :

Theorem 7. *Let $\varphi \in C^{1,\delta}(V, V \otimes V^\star)$ with $(1+\delta)\gamma > 1$ and $\gamma < \frac{1}{2}$. Then each stochastic integral in the left column of the table has a continuous version which equals almost surely the corresponding integral on the right. Moreover, by the relationship between Itô and Stratonovich integration:*

$$\delta I^\mu_{\mathit{Itô}} + \frac{g^{\kappa\nu}}{2} \int_s^t \partial_\kappa \varphi(X_u)^\mu_\nu \, du = \delta I^\mu_{\mathit{Strat.}}\,,$$

we have:

$$\delta I^\mu_{\mathit{rough},st} + \frac{g^{\kappa\nu}}{2} \int_s^t \partial_\kappa \varphi(X_u)^\mu_\nu \, du = \delta J^\mu_{st}.$$

References

[1] Gubinelli M., *Controlling rough paths.* J. Funct. Anal., 216 (1) (2004), pp. 86??140;

[2] S. Kwapień Rosiński, *Sample Hölder continuity of stochastic processes and majorizing measures.* Seminar on Stochastic Analysis, Random Fields and Applications IV, (2004), pp. 155-163;

[3] D.W. Stroock *Probability Theory* (3rd Edition), Cambridge University Press, Cambridge (1993).

Gianmarco Brocchi, University of Bonn
email: **gianmarcobrocchi@gmail.com**

4 Ramifications of Rough Paths

after Massimiliano Gubinelli [4]
A summary written by Robert A. Crowell

Abstract

The work of [4] is surveyed. It extends Chen's multiplicative property by indexing iterated integrals with labeled trees. These trees are enriched with algebraic structure which involves the Dürr-Connes-Kreimer co-product encoding the combinatorics of trees. The theory is a natural setting for a non-geometric theory of rough paths.
All errors are mine. No claim on originality is made.

4.1 Introduction

We shall discuss some parts of the work in [4]. The theory of abstract integration against irregular paths of [3] is further algebraicized. Iterated integrals are indexed by decorated trees on which a rich algebraic structure can be introduced. This structure involves the Dürr-Connes-Kreimer co-product which encodes the combinatorics of labeled trees. In this way an algebraically powerful and notationally convenient apparatus can be obtained providing a solid basis for a non-geometric theory of rough paths. In particular Chen's multiplicative property [1], a basis of Lynons' [5] theory, naturally extends within this structure. This will lead to the concept of branched rough paths, and extends the theory of integration against irregular paths of [3] to Hölder-exponents $\gamma > 0$.

4.2 Rooted trees and iterated Integrals

We build a bridge between iterated integrals and rooted decorated trees. In the following we outline how to enrich the sets of rooted trees and increment-functions with algebraic structure and how to construct our homomorphism.

Trees A *rooted tree* is a finite, cycle-free graph with a distinguished vertex. Given a finite set $\mathcal{L} = \{a_1, \dots, a_k\}$, a $\mathcal{L}$*-labeling* (or a *decoration*) is an association of a label in $\mathcal{L}$ to every vertex of the rooted tree.

It is useful to observe the following iterative procedure of recursively growing trees. Given $\tau_1, \dots, \tau_k$ labeled rooted trees and a label $a \in \mathcal{L}$, define

$$[\tau_1, \dots, \tau_k]_a$$

as the tree obtained by attaching to a vertex labeled a and declared as the new root the trees $\tau_1, \dots, \tau_k$. Any tree can be recursively grown using only the set of labeled vertices and successively applying $[-]_-$.

Denoting by $\mathcal{T}_\mathcal{L}$ the set of all $\mathcal{L}$-decorated trees and by $\mathbf{1}$ be the empty tree, we obtain the $\mathbb{R}$-Algebra of *polynomials on decorated trees*,

$$\mathcal{AT}_\mathcal{L} = \langle \{\mathbf{1}\} \cup \mathcal{T}_\mathcal{L} \rangle_{\mathbb{R}-\mathbf{Alg}}$$

Its elements are finite formal sums of formal monomials weighted by coefficients in $\mathbb{R}$. A monomial of decorated trees $\tau_1 \cdots \tau_k$ is called a *forest*. We shall denote the set of forests by $\mathcal{F}_\mathcal{L}$.

The algebra structure can be dualized to obtain a co-algebra, its co-product $\nabla : \mathcal{AT}_\mathcal{L} \to \mathcal{AT}_\mathcal{L} \otimes \mathcal{AT}_\mathcal{L}$ will play a key role in the constructions below. It acts on forests as $\Delta(\tau_1 \cdots \tau_k) = \Delta(\tau_1) \cdots \Delta(\tau_k)$ and is defined on generators by $\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$ and recursively on trees $\tau \in \mathcal{T}_\mathcal{L}$ via

$$\Delta(\tau) := \mathbf{1} \otimes \tau + \sum_{a \in \mathcal{L}} (B_+^a \otimes \mathsf{id})[\Delta(B_-^a(\tau))].$$

For $a \in \mathcal{L}$ we define the map $B_+^a(\mathbf{1}) = \bullet_a$ and $B_+^a(\tau_1 \cdots \tau_k) = [\tau_1, \dots, \tau_k]_a$. We shall also need its inverse B_-^a, which removes the root if it is labeled by a and erases the entire tree otherwise, *i.e.*

$$B_-^a(B_+^b(\tau_1 \cdots \tau_k)) = \begin{cases} \tau_1 \cdots \tau_k & \text{if } a = b, \\ \mathbf{1} & \text{if } a \neq b. \end{cases}$$

These dual structures satisfy the compatibility conditions of a bi-algebra, [2].

The co-product encodes the combinatorics of the tree, *viz.* By an *admissible cut* c of a tree τ we mean detaching a set of branches from the tree. Let $\mathsf{C}(\tau)$ be the set of admissible cuts for τ. Given such a cut c, denote by $\mathsf{R}_c(\tau) \in \mathcal{T}_\mathcal{L}$ the remaining subtree and by $\mathsf{P}_c(\tau) \in \mathcal{F}_L$ the forest of detached and newly planted branches. In [2] it is shown that we have the explicit description of the co-product in terms of cuts

$$\Delta(\tau) = \mathbf{1} \otimes \tau + \tau?? \otimes \mathbf{1} + \sum_{c \in \mathsf{C}(\tau)} \mathsf{R}_c(\tau)?? \otimes \mathsf{P}_c(\tau)$$

Thus the co-product consists of the collection of all subtrees and forests stemming form systematically decomposing the tree using cuts (even the two trivial ones). Below we shall only be using the reduced co-product

$$\Delta'(\tau) = \Delta(\tau) - \mathbf{1} \otimes \tau - \tau \otimes \mathbf{1}$$

It is straight forward using the definitions to verify that

$$\Delta'(\tau) = \sum_{a \in \mathcal{L}} \bullet_a \otimes B_-^a(\tau) + \sum_{a \in \mathcal{L}} (B_+^a \otimes \mathsf{id}) \Big[\Delta' \Big(B_-^a(\tau) \Big) \Big] \tag{1}$$

The reduced co-product will be used to break up tree-polynomials into small pieces for which we are given certain data.

Increments Given $T > 0$ and $k \in \mathbb{N}$, denote by $\mathcal{C}_k$ the set of continuous functions $g : [0,T]^k \to \mathbb{R}$, such that $g(t_1, \ldots, t_k) = g_{t_1,\ldots t_k} = 0$ whenever $t_i = t_{i+1}$ for $0 \leq i \leq k-1$. We call such a function a *k-increment* and define $\mathcal{C}^* = \bigoplus_{k \geq 0} \mathcal{C}_k$. The vector-space of k-increments can be turned into an algebra by declaring $(f \circ g)_{t_1 \cdots t_k} = f_{t_1 \cdots t_k} g_{t_1 \cdots t_k}$ as inner product.

By a chain-complex constructions as in [3] we obtain the co-boundary

$$\delta : \mathcal{C}_k \to \mathcal{C}_{k+1} \qquad g \mapsto (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}$$

and the associated long exact sequence (C^*, δ). Let us agree to denote by $\mathcal{ZC}_k = \mathcal{C}_k \cap \mathsf{Ker}(\delta)$ the space of co-cycles.

Of particular interest is $\mathcal{C}_2$. Its elements include iterated integrals of smooth functions on $[0,T]$. Evidently for $f \in \mathcal{C}_1^\infty$ and $h \in \mathcal{C}_2$, we have

$$(s,t) \mapsto \mathcal{I}(df\ h)_{st} := \int_s^t h_{us} df_u \in \mathcal{C}_2.$$

The following action of δ on the integral will be used throughout. Equation (2) will be used to give an axiomatic definition of integrals.

Lemma 1. *Let $h \in \mathcal{C}_2$ such that $\delta h_{tus} = \sum_{i=1}^N h_{tu}^{1,i} h_{us}^{2,i}$ for some $N \in \mathbb{N}$, $h^{1,i}, h^{2,i} \in \mathcal{C}_2$ and let $x \in C_1^\infty([0,T], \mathbb{R})$. Then*

$$\delta \mathcal{I}(dx\ h)_{tus} = \mathcal{I}(dx)_{tu} h_{us} + \sum_{i=1}^N \mathcal{I}(dh\ h^{1,i})_{tu} h_{us}^{2,i} \tag{2}$$

Mapping trees to integrals As we have the structure of an $\mathbb{R}$-Algebra on trees and 2-increments, we can extend maps prescribed on generators.

Let $\mathcal{L} = \{1, \dots, d\}$ and suppose we are given a vector $\{x^i\}_{i\in\mathcal{L}}$ of elements in $\mathcal{C}_1^\infty$. Recall that any tree can be grown using only labeled nodes and $[-]_-$. Define the map $X : \mathcal{T}_\mathcal{L} \to C([0,T]^2, \mathbb{R})$ by

$$X^{\bullet_a}_{st} = \int_s^t dx^a_u = \delta x^a_{st} \qquad X^{[\tau_1 \cdots \tau_k]_a}_{st} = \int_s^t \left(\prod_{i=1}^k X^{\tau_i}_{us} \right) dx^a_u \tag{3}$$

and extend it to a morphism of algebras $\mathcal{AT}_\mathcal{L} \to \mathcal{C}_2^+$ (here $\mathcal{C}_2^+$ is the previously defined algebra freely adjoined unit).

4.3 The multiplicative property

With the structure outlined above we have a powerful algebraic toolbox and a notationally convenient apparatus to systematically generalize Chen's multiplicative property of iterated integrals.

Integration on a sub-algebra Induced by the family $\{x^a\}_{a\in\mathcal{L}}$ are the integration maps $\{I^a : \mathcal{C}_2 \to \mathcal{C}_2\}_{a\in\mathcal{L}}$ defined via $I^a(h) = \mathcal{I}(dx^a\ h)$. The following fundamental commutativity relation is a direct consequence of the definitions,

$$I^a X^\sigma = X^{B^a_+(\sigma)}?? \quad \text{for all } \sigma \in \mathcal{AT}_\mathcal{L}. \tag{4}$$

Hence X maps the family $\{I^a\}_a$ onto $\{B^a_+\}_a$. and thus we can represent integration on the sub-algebra $\mathcal{A}_X \subset \mathcal{C}_2^+$ generated by $\{X^\tau\}_{\tau\in\mathcal{T}_\mathcal{L}}$ via B^a_+. We also note that the functions $\{B^a_+\}_a$ enter the definition of the co-product.

Chen's multiplicative property As a first step towards understanding the interplay of the algebraic structure with the co-boundary and the co-product it is helpful to examine a simple case explicitly.

From the family $\{x^a\}_{a\in\mathcal{L}}$ define iterated integrals recursively as follows.

$$\mathcal{I}(dx^{a_1}\ dx^{a_2} \cdots dx^{a_n}) = \mathcal{I}(dx^{a_1}\ \mathcal{I}(dx^{a_2}\ dx^{a_3} \cdots dx^{a_n}))$$

Via Lemma 1 we recover Chen's multiplicative property

$$\delta\mathcal{I}(dx^{a_1} \cdots dx^{a_n})_{stu} = \sum_{k=1}^{n-1} \mathcal{I}(dx^{a_1} \cdots dx^{a_k})_{st} \mathcal{I}(dx^{a_{k+1}} \cdots dx^{a_n})_{tu}$$

The sub-algebra $\mathcal{A}_X$ contains the usual iterated integrals, which correspond to trees of the form $[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}$.

$$\mathcal{I}(dx^{a_1}\cdots dx^{a_n}) = I^{a_1}\cdots I^{a_{n-1}}(\delta x^{a_n}) = X^{B_+^{a_1}\cdots B_+^{a_{n-1}}\bullet_{a_n}} = X^{[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}}$$

We shall call such non-branching trees sticks. Their combinatorics are particularly simple; Any non-trivial cut of a stick $\sigma = [\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_1}$ breaks it into two pieces, so that

$$\Delta'(\sigma) = \sum_{k=1}^{n-1}[\cdots[\bullet_{a_k}]_{a_{k-1}}\cdots]_{a_1}\otimes[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_{k+1}}$$

Hence using the extension of X to the tensor product via the exterior product $\mathcal{C}_2\otimes\mathcal{C}_2\to\mathcal{C}_3$, $f_{st}\otimes g_{uv}\mapsto f_{st}g_{tu}$, we get

$$X^{\Delta'(\sigma)} = \sum_{k=1}^{n-1}X^{[\cdots[\bullet_{a_k}]_{a_{k-1}}\cdots]_{a_1}}X^{[\cdots[\bullet_{a_n}]_{a_{n-1}}\cdots]_{a_{k+1}}}$$

and thus $\delta X^\sigma = X^{\Delta'(\sigma)}$ via Chen's property.

Generalization of Chen's multiplicative property The above illustrates a more general principle: the relation between the co-algebra structure on trees and its interplay with the co-boundary operator acting on integrals. The following extension to arbitrary tree polynomials of the relation derived above is a fundamental.

Theorem 2. *The map $X:\mathcal{AT}_\mathcal{L}\to\mathcal{C}_2$ satisfies the following algebraic relation*

$$\delta X^\sigma = X^{\Delta'(\sigma)} \quad \text{for all } \sigma\in\mathcal{AT}_\mathcal{L} \tag{5}$$

where Δ' is the reduced co-product.

4.4 Branched rough paths

It is evident from the proofs in [4], that relation (4) and the tree multiplicative property of Theorem 2 hold more generally for any family $\{X^\tau\}_{\tau\in\mathcal{T}_\mathcal{L}}$ stemming from linear maps $\{I^a:\mathcal{C}_2\to\mathcal{C}_2\}_{a\in\mathcal{L}}$ satisfying certain properties. This can be used to give an axiomatic definition of integrals.

An axiomatic definition of integrals We take linearity of classical integrals and the properties of Lemma 1 as a starting point.

Definition 3. *Call a linear map $I : \mathcal{D}_I \to \mathcal{D}_I$ on a sub-algebra $\mathcal{D}_I \subset \mathcal{C}_2^+$ containing the unit $e \in \mathcal{C}_2$ an* integral *if is satisfies the following properties.*

1. $I(hf)_{ts} = I(h)_{ts} f_s$, *for all $h \in \mathcal{D}_I, f \in \mathcal{C}_1$ where $(hf)_{ts} = h_{ts} f_s$,*
2. $\delta I(h)_{tus} = I(e)_{tu} h_{us} + \sum_{i=1}^{N} I(h^{1,i})_{tu} h^{2,i}_{us}$
whenever $h \in \mathcal{D}_I$ and $\delta h_{tus} =?? \sum_{i=1}^{N} h^{1,i}_{tu} h^{2,i}_{us}$ for some $n \in \mathbb{N}$, $h^{1,i} \in \mathcal{D}_I$.

As in (3) one can associate to a given family of integrals $\{I^a\}_{a \in \mathcal{L}}$ in the sense of Definition 3, a family of functions $\{X^\tau\}_{\tau \in \mathcal{F}_\mathcal{L}}$. The extension $\mathcal{AT}_\mathcal{L} \to \langle \{X^\tau\}_{\tau \in \mathcal{F}_\mathcal{L}} \rangle$ satisfies (4) and the multiplicative property, Theorem 2.

Regularity conditions and extension We can now introduce regularity conditions on the map X. Once imposed, they will allow us to extend a *finite* family $\{X^\tau\}$ uniquely to the whole algebra of tree-polynomials satisfying certain bounds. This will lead us to the concept of branded rough paths.

For $\mu > 0$ we introduce the Hölder-like norms on subspaces of $\mathcal{C}_2$ and $\mathcal{C}_3$ respectively. For $f \in \mathcal{C}_2$ and $h \in \mathcal{C}_3$ we let

$$\|f\|_\mu := \sup_{s \neq t, s,t \in [0,T]} \left\{ \frac{f_{st}}{|s-t|^\mu} \right\} \text{ and } \|h\|_{\gamma,\rho} := \sup_{s,u,t \in [0,T]} \left\{ \frac{|h_{tus}|}{|u-s|^\gamma |t-u|^\rho} \right\}$$

and define

$$\|h\|_\mu := \inf_{0 < \rho_i < \mu} \left\{ \sum_{i=1}^{N} \|h_i\|_{\rho_i, \mu - \rho_i} \ : \ h = \sum_{i=1}^{N} h_i, \ h_i \in \mathcal{C}_3, \ N \in \mathbb{N} \right\}.$$

We then obtain the Banach-space $\mathcal{C}_2^\mu := \{f \in \mathcal{C}_2 : \|f\|_\mu < \infty\}$ and similarly $\mathcal{C}_3^\mu := \{f \in \mathcal{C}_3 : \|f\|_\mu < \infty\}$. Finally we define $\mathcal{C}_k^{1+} = \cup_{\mu > 1} \mathcal{C}_k^\mu$.

The regularity of functions is related to the size of trees as follows. Given $\gamma \in (0,1]$ define the function q_γ on trees as $q_\gamma(\tau) = 1$ whenever $|\tau| \leq 1/\gamma$ and for $|\tau| > 1/\gamma$ set

$$q_\gamma(\tau) = \frac{1}{2^{\gamma|\tau|} - 2} \sum q_\gamma(\tau^{(1)}) q_\gamma(\tau^{(2)})$$

The sum and the splitting of trees arise from the reduced co-product (1). On a forest $\tau = \tau_1 \cdots \tau_k$, we set $q_\gamma(\tau) = q_\gamma(\tau_1) \cdots q_\gamma(\tau_k)$.

Definition 4. *Let $\gamma > 0$. We call a morphism of algebras $X : \mathcal{AT}_{\mathcal{L}} \to \mathcal{C}_2$ a* γ-branched rough path *(γ-BRP) if it satisfies (5) and*

$$\|X^\tau\|_{\gamma|\tau|} \leq BA^{|\tau|} q_\gamma(\tau), \quad \text{for all } \tau \in \mathcal{F}_{\mathcal{L}}$$

and constants $B \in [0,1]$ and $A?? \geq 0$.

Recall from [3] the sewing-map $\Lambda : \mathcal{ZC}_3^{1+} \to \mathcal{C}_2^{1+}$, where it was used in the splitting of a short exact sequence. In particular the co-boundary δ is its inverse. With its help, we can define $\gamma-$BRP from a finite set of data.

Let $n \in \mathbb{N}$ and $\gamma \in (0,1)$ be such that $\gamma(n+1) > 1$. Suppose we are given a morphism $X : \mathcal{A}_n\mathcal{T}_{\mathcal{L}} \to \mathcal{C}_2$ on the sub-vectorspace of tree-polynomials with degree at most n that satisfies (5) and such that

$$\|X^\tau\|_{\gamma|\tau|} \leq BA^{|\tau|} q_\gamma(\tau)$$

for all $\tau \in \mathcal{T}_{\mathcal{L}}{}^n$ and constants $\gamma > 0$, $A \geq 0$, $B \in [0,1]$. Then this map can be extended uniquely to a γ-BRP with the same bounds. The key in the construction is that X^τ will be in the domain of Λ for large enough trees. This construction allows us to go below $\gamma < 1/3$, *cf.* [3], [4].

Rough paths can be used to define *controlled paths*. These are paths that constitute a natural space to solve rough differential equations, i.e. equations of the form

$$\delta y = \sum_{a \in \mathcal{L}} I^a(f_a(y)), \qquad y_0 =??\eta \in \mathbb{R}^k$$

Where $\{f_a \in \mathcal{C}(\mathbb{R}^k, \mathbb{R}^k),\ a \in \mathcal{L}\}$ is a family of sufficiently regular vector-fields and $\{I^a\}$ a family of integral maps that define a $\gamma-$BRP.

In particular controlled paths constitute an algebra that is closed under integration against rough paths and the application of sufficiently regular functions.

References

[1] Chen, Kuo-Tsai, *Iterated Path Integrals.* Bulletin of the American Mathematical Society 83.5 (1977): 831-879.

[2] Connes Alain and Dirk Kreimer, *Hopf Algebras, Renormalization, and Noncommutative Geometry*, Comm. Math. Phys. 199 (1998), 203-242.

[3] Gubinelli, Massimiliano: *Controlling Rough Paths*, Journal of Functional Analysis, Volume 216, Issue 1, 2004, Pages 86-140, doi:10.1016/j.jfa.2004.01.002.

[4] Gubinelli, Massimiliano: *Ramification of Rough Paths*, Journal of Differential Equations, Volume 248, Issue 4, 2010, Pages 693-721, ISSN 0022-0396, doi:10.1016/j.jde.2009.11.015.

[5] Lyons, Terry J. *Differential Equations Driven by Rough Signals*. Rev. Mat. Iberoamericana 14.2 (1998).

Robert Alexander Crowell, University of Bonn
email: **crowellr@uni-bonn.de**

5 Integrability and tail estimates for Gaussian rough differential equations

after T. Cass, C. Litterer, and T. Lyons [1]
A summary written by Luigi Borasi

Abstract

We discuss the results presented in [1] where the authors proved that the Jacobian of the solution flow for stochastic differential equations driven by a wide class of Gaussian processes (including fractional Brownian motion with Hurst parameter $H > 1/4$) has finite moments. Explicit bounds are given.

5.1 Introduction

The article is motivated by the study of SDEs of the form

$$dY_t = V(Y_t)dX_t, \quad Y_0 = y_0 \tag{1}$$

which is driven by a Gaussian process X which is not necessarily a semimartingale. In particular the case of a fractional Brownian motion (fBm) with Hurst parameter $H > 1/4$ is given special attention.

A natural question in connection to equation (1) regards the existence and smoothness of the density of the solution. Existence of the density was established by [3]. While [8, 2] smoothness of the density was established in the regime $H > 1/2$ (therefore whithin the realm of classical Young integration).

The main problem in extending the result in [8, 2] to the case of more general Gaussian processes is the need for sharp estimates on the integrability of the Jacobian

$$J^{\mathbf{X}}_{t\leftarrow 0}(\cdot) \equiv DU^{\mathbf{X}}_{t\leftarrow 0}(\cdot)$$

where $U^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ is the flow associated with the solution of (1).

The main result in the summarized article[2] is that the logarithm of the Jacobian has a tail that decays faster than expenentially, more precisely it is shown that for any $r < r_0 \in (1, 2]$

$$\mathbb{P}(\log[|J^{\mathbf{X}}_{\cdot\leftarrow 0}(y_0)|_{p\text{-var};[0,T]}] > 0) \lesssim \exp(-x^r). \tag{2}$$

[2]It is worth mentioning that this result also appears now with a slightly altered, somewhat shorter proof in [4].

The constant r_0 is described in terms of the regularity properties of the Gaussian path. From (2) the existence of moments of all orders for $J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$ follows easily.

It is worth mentioning two of the main applications. First, these results are needed for extending the work of [6] and [8] on the ergodicity of non-Markovian systems Second, they are important in the Malliavin calculus proof of Hörmander-type theorems. In this context we recall the recent work of Hu and Tindel [9] (extending Norris' lemma for fBm with $H > 1/3$) , and of Hairer and Pillai [7] (proving Hörmander-type theorems for a general class of RDEs under the assumption that the Jacobian has finite moments of all orders).

5.2 Notation

We shall employ the following notation. Let $C(I, \mathbb{R}^d)$ denote the space continuous $\mathbb{R}^d$-valued paths x parametrixed by time on a compact interval I. We use the shorthand $x_{s,t} := x(s) - x(t)$ when $x \in C(I, \mathbb{R}^d)$. For $p \geq 1$ we will use the p-variation semi-norm

$$|x|_{p-var;I} := \left(\sup_{D[I]=(t_j)} \sum_{j:t_j \in D[I]} |x_{t_j,t_{j+1}}|^p \right)^{1/p}$$

The space of weakly geometric p-rough pats will be denoted by $WG\Omega_p(\mathbb{R}^d)$. For $\mathbf{x} \in WG\Omega_p(\mathbb{R}^d)$ we let

$$\|\mathbf{x}\|_{p\text{-var};[0,T]} := \left(\sum_{i=1}^{\lfloor p \rfloor} \sup_{D=(t_j)} \sum_{j:t_j \in D} |\mathbf{x}^i_{t_j,t_{j+1}}|^{p/i}_{(\mathbb{R}^d)^{\otimes i}} \right)^{1/p}.$$

5.3 Translated rough paths

The next important lemma shows how to control the p-variation of the translated rough path by the sum of the p-variation of the rough path itself and the q-variation of the path by which we translate.

Lemma 1 (Lemma 3.1)**.** *Let $1 \leq p < 4$. Suppose that $\mathbf{x}$ is a weakly geometric p-rough path parametrised over a compact interval I. Let h be a path in*

$C^{q\text{-var}}(I, \mathbb{R}^d)$ *where* $1/q + 1/p > 1$. *If* $T_h\mathbf{x}$ *denotes the translated rough path, then for any* $[s,t] \subset I$ *we have the estimate*

$$\|T_h\mathbf{x}\|^p_{p\text{-var};[s,t]} \leq C_{p,q}[\|\mathbf{x}\|^p_{p\text{-var};[s,t]} + |h|^q_{q\text{-var};[s,t]}].$$

The constant $C_{p,q}$ *can be given explicitly.*

5.4 Deterministic estimates for solutions to RDEs

A novel contribution of this article is the definition of the following object which proves useful in a variety of problems.

Definition 2 (The greedy sequence). *Let* $\mathbf{x} \in WG\Omega_p(\mathbb{R}^d)$ *be parametrized over a compact* I. *For* $\alpha > 0$ *we define a nondecreasing sequence* $(\tau_i(\alpha, p, \mathbf{x}))_{i=0}^\infty = (\tau_i(\alpha))_{i=0}^\infty \subset I$ *by*

$$\begin{aligned} \tau_0(\alpha) &= \inf I, \\ \tau_{i+1}(\alpha) &= \sup I \wedge \inf\{t : \|\mathbf{x}\|^p_{p\text{-var};[\tau_i,t]} \geq \alpha, \tau_i(\alpha) < t \leq \sup I\}. \end{aligned} \tag{3}$$

Moreover we introduce the function $N_{\alpha,I,p} : WG\Omega_p(\mathbb{R}^d) \to \mathbb{R}_+$ *given by*

$$N_{\alpha,I,p}(\mathbf{x}) := \sup\{n \in \mathbb{N} \cup \{0\} : \tau_n(\alpha) < \sup I\}.$$

Comments. *1. Intuitively, for* $\tau_i(\alpha) < \sup I$ *and* $\|\mathbf{x}\|_{p\text{-var};[\tau_i,t]} \geq \alpha$, $\tau_{i+1}(\alpha)$ *is the first time* $\|\mathbf{x}\|_{p\text{-var};[\tau_i,t]}$ *reaches* α.

2. The number of distinct terms in the sequence $(\tau_i(\alpha))_{i=0}^\infty$ *is given by* $N_{\alpha,I,p}(\mathbf{x}) + 1$.

3. It can be shown that the function $N_{\alpha,I,p}$ *is well defined [Lemma 4.9 in [1]] and the sequence* $(\tau_i(\alpha))_{i=0}^{N_{\alpha,I,p}(\mathbf{x})+1}$ *is a partition of* I *(for* $\mathbf{x} \in WG\Omega_p(\mathbb{R}R^d)$*) [Corollary 4.10 in [1]].*

Proposition 3 (Corollary 4.6 + Proposition 4.11). *Let* $p \geq 1$ *and suppose* $\mathbf{x}$ *is a path in* $WG\Omega_p(\mathbb{R}^d)$ *parametrized over the compact interval* I. *Then for every* $\alpha > 0$ *we have the estimate*

$$\begin{aligned} &|J^{\mathbf{x}}_{\cdot\leftarrow 0}(y_0)|_{p\text{-var};[0,T]} \\ &\quad \leq C|V|_{\text{Lip-}\gamma}\|\mathbf{x}\|_{p\text{-var};[0,T]} \exp[C\max(1, \alpha|V|^p_{\text{Lip-}\gamma})(2N_{\alpha,[0,T],p}(\mathbf{x}) + 1)]. \end{aligned} \tag{4}$$

In the following we will take $\mathbf{x} = \mathbf{X}$ to be a Gaussian rough path. Then the tail of the Jacobian will be studied via the tail of $N_{\alpha,I,p}(\mathbf{X})$.

5.5 Gaussian rough paths

In this section[3] and the next we let $(\mathcal{W}, \mathcal{H}, \mu)$ be an abstract Wiener space and $(X_t)_{t\in I} = (X_t^1, \ldots, X_t^d)_{t\in I}$ be a continuous, mean-zero Gaussian process with independent and identically distributed components on $(\mathcal{W}, \mathcal{H}, \mu)$ each one with covariant function $R(s,t) = \mathbb{E}[X_t^1 X_s^1]$, $s < t$. Moreover we use the notation

$$R\left(\begin{smallmatrix} s,t \\ s',t' \end{smallmatrix}\right) := \mathbb{E}[(X_t^1 - X_s^1)(X_{t'}^1 - X_{s'}^1)], \quad s < t,\ s' < t'.$$

Condition 1 (Finite ρ-variation)**.** There exists $\rho \in [1,2)$ such that R has finite ρ-variation, i.e.

$$V_\rho(R; I \times I) := \left(\sup_{(t_i),(t'_j) \in \mathcal{D}(I)} \sum_{i,j} \left| R\left(\begin{smallmatrix} t_i, t_{i+1} \\ t'_j, t'_{j+1} \end{smallmatrix}\right) \right|^\rho \right)^{1/\rho} < +\infty. \tag{5}$$

The following lemma show that under certain conditions $\mathbf{X}(\omega+h)$ is equal to $T_h\mathbf{X}(\omega)$ for all $h \in \mathcal{H}$ on a set of μ-full measure.

Lemma 4 (Lemma 5.4)**.** *With $(X_t)_{t\in I}$ as above, assume that X has a natural lift to a geometric p-rough path. Assume further that for some $q \geq 1$ such that $1/p + 1/q > 1$, we have $\mathcal{H} \hookrightarrow C^{q\text{-var}}(I, \mathbb{R}^d)$. Then there exists a measurable subset $E \subset \mathcal{W}$ with $\mu(E) = 1$, such that for all $\omega \in E$, we have*

$$T_h\mathbf{X}(\omega) \equiv \mathbf{X}(\omega + h) \quad \textit{for all } h \in \mathcal{H}.$$

The following corollary shows that in the examples of Gaussian processes which interest us the most we indeed have that X lifts to path in $G\Omega_p(\mathbb{R}^d)$ and $\mathcal{H}$ continuously embed in $C^{q\text{-var}}(I, \mathbb{R}^d)$.

Corollary 5 (Corollary 5.5)**.** *Let $(X_t)_{t\in I}$ be such that the covariance function R satisfies Condition 1. Suppose at least one of the following holds:*
1. For some $\rho \in [1, 3/2)$ the covariance function of X has finite ρ-variation in the sense of Condition 1;
2. X is a fractional Brownian motion for H in $(1/4, 1/2)$.

Then there exist real numbers p, q such that the following statements are true simultaneoously
1. X has a natural lift to a geometric p-rough path;
2. $\mathcal{H} \hookrightarrow C^{q\text{-var}}(I, \mathbb{R}^d)$ where $1/p + 1/q > 1$.

[3]For more information on this section cfg. [4] and reference in [1].

5.6 The tail behaviour of $N_{\alpha,I,p}(X(\cdot))$ via Gaussian isoperimetry

Theorem 6 (Borell's isoperimetric inequality). *Let $(\mathcal{W},\mathcal{H},\mu)$ be an abstract Wiener space and $\mathcal{K}$ denote the unit ball in $\mathcal{H}$. Suppose A is a Borell subset of $\mathcal{W}$ such that $\mu(A) \geq \Phi(a)$ for some real number a. Then for every $r \geq 0$,*

$$\mu_*(A + r\mathcal{K}) \geq \Phi(a + r),$$

where μ_* is the inner measure of μ, and Φ denotes the standard normal cumulative distribution function.

The next Proposition is cucial. It allows for the application of Borell's inequality to control the tail of the random variable $N_{\alpha,I,p}(\mathbf{X}(\omega))$.

Proposition 7 (Proposition 6.2). *Let (X_t) as in the previous section. Suppose that p, q are such that $1 \leq p < 4$ and $1/p + 1/q > 1$. Assume furthermore that 1. X has a natural lift to a geometric p-rough path $\mathbf{X}$; 2. $\mathcal{H} \hookrightarrow C^{q\text{-var}}(I, \mathbb{R}^d)$. Then there exists a set $E \subset \mathcal{W}$, of μ-full measure, with the following property: for all ω in E, h in $\mathcal{H}$, and $\alpha > 0$,*

$$\text{if} \quad \|\mathbf{X}(\omega - h)\|_{p\text{-var};I} \leq \alpha, \quad \text{then} \quad |h|_{q\text{-var};I} \geq \alpha N_{\tilde{\alpha}^p,I,p}(\mathbf{X}(\omega))^{1/q},$$

where $C_{p,q}$ is the constant in Lemma 1 and $\tilde{\alpha} = (2C_{p,q})^{1/p}\alpha$.

Theorem 8 (Theorem 6.5: **Moment estimates on the Jacobian**). *Let $(X_t)_{t\in I}$ be as above. Let $\rho \in [1, 3/2]$, $p \in (2\rho, 3)$, and $\gamma > p$. Suppose Condition 1. in the previous section holds.*

Then 1. X lifts to a geometric p-rough path $\mathbf{X}$, and for any collection of Lip-γ vector fields $V = (V^1, \ldots, V^d)$ on $\mathbb{R}^e$ the RDE

$$dY_t = V(Y)d\mathbf{X}_t, \quad Y(0) = y_0$$

has a unique solution. 2. The flow $U^{\mathbf{X}(\omega)}_{t\leftarrow 0}(\cdot)$ induced by the solution ot this RDE is differentiable. 3. The Jacobian $J^{\mathbf{X}}_{t\leftarrow 0}(\omega)(y_0)\cdot a := \frac{d}{d\epsilon}\big|_{\epsilon=0} U^{\mathbf{X}(\omega)}_{t\leftarrow 0}(y_0+\epsilon a)$ satisfies

$$\exp[(\log |J^{\mathbf{X}}_{t\leftarrow 0}(\cdot)(y_0)|)^r] \in \bigcap_{q>0} L^q(\mu)$$

for all $y_0 \in \mathbb{R}^e$ and all $r < 2/rho$.

The above result applies to fractional Brownian motion with $H > 1/3$. It turns out that the hypothesis (on the specific embedding) can be leveraged and a similar result can be derived also for $H > 1/4$ (Theorem 6.6 in [1]).

References

[1] Cass, T., Litterer, C., and Lyons, T. (2013), *Integrability and tail estimates for Gaussian rough differential equations*. Ann. Prob. Vol. 41 No. 4 , 3026-3050.

[2] Baudoin, F. and Hairer, M. (2007). A version of Hörmander's theorem for the fractional Brownian motion. *Probab. Theory Related Fields* **139** 373-395

[3] Cass, T. and Friz, P. (2010). Densities for rough differential equations under Hörmander's condition *Ann. of Math.* (2) **171** 2115-2141.

[4] Friz, P. and Hairer, M. (2014), *A Course on Rough Paths: With an introduction to regularity structures*. Springer.

[5] Friz, P. and Victoir, N. (2010). *Multidimensional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge Studies in Advanceed Mathematics* **120**. Cambridge Univ. Press, Cambridge.

[6] Hairer, M. and Mattingly, J. C. (2006). Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Ann. of Math.* (2) **164** 993-1032.

[7] Hairer, M. and Pillai, N. (2011). Regularity of laws and ergodicity of hypoelliptic SDEs driven by rough paths. Available at arXiv:1104.5218v1

[8] Hairer, M. and Pillai, N. (2011). Ergodicity of hypoelliptic SDEs driven by fractional Brownian motion *Ann. Inst. Henri Poincaré Probab. Stat.* **47** 601-628

[9] Hu, Y. and Tindel, S. (2011). Smooth density for some nilpotent rough differential equations. Available at arXiv:1104.1972

Luigi Borasi, University of Bonn
email: **borasi@iam.uni-bonn.de**

6 Inverting the signature of a path

after T. Lyons and W. Xu [1]
A summary written by João Pedro Ramos.

Abstract

We give an explicit algorithm to recover – up to small Lipschitz-norm error – a path given its set of signatures. Moreover, we give an upper bound on the number of elements from the signature needed to approximate the path up to order k, k sufficiently large.

6.1 Introduction

Let $\gamma : [0,1] \to \mathbb{R}^d$ be a continuously differentiable path at *natural parametrization*, that is, such that, if $\gamma = (\gamma_1, \cdots, \gamma_d)$, then

$$\sum_{i=1}^{d} |\gamma_i'(t)| = L = \text{const.},$$

for all $t \in [0,1]$. An object of extreme interest related to these paths, and moreover to the analysis of rough paths, is the *signature*. Before describing what a signature is, we mention that a *word of length k* is a sequence of k elements, where each of those is taken from $\{e_1, ..., e_d\}$, where e_i is the vector consisting of 1 on the i-th position, and zero on every other one. For example, if $d = 2$, $\{xy, xx, yx, yy\}$ form the collection of all length 2 words. We denote the *concatenation* of two words $w_1 = e_{i_1} \cdots e_{i_k}, w_2 = e_{j_1} \cdots e_{j_s}$ as the word

$$w_1 * w_2 = e_{i_1} \cdots e_{i_k} e_{j_1} \cdots e_{j_s}.$$

That being defined, we define the *signature of the word $w = e_{i_1} \cdots e_{i_k}$ by* γ as the number

$$C_\gamma(w) = \int_{0<t_1<\cdots<t_k<1} \gamma_{i_1}'(t_1) \cdots \gamma_{i_k}'(t_k)\, dt_1 \cdots dt_k.$$

Of course, the *signature of the path* γ is just the collection of the signatures for all possible words, which can be concisely defined via the formal sum

$$X(\gamma) = \sum_{k \geq 0} \sum_{|w|=k} C_\gamma(w) w.$$

One natural question to ask is then if, given the signature of a path $X(\gamma)$, we can recover the path itself. Other references [2, 3] investigated the problem in, respectively, the setting of finite length paths and geometric rough paths. Nevertheless, there were no explicit reconstruction procedures before [1]. Therefore, the main result is an explicit way to obtain a piecewise linear path approximating the original one in Lipschitz norm directly from the signatures up to a certain level. Explicitly:

Theorem 1. *Let $d = 2$. Given the signatures of γ for words w such that $|w| \leq 8k^3 \log k + k$, we may find a path $\tilde{\gamma}$, which consists of k linear parts concatenated, and*

$$\sup_{t\in[0,1]} |\gamma'(t) - \tilde{\gamma}'(t)| \leq C\eta_k,$$

where $C > 0$ depends only on (the signature of) γ, and $\eta_k \to 0$ as $k \to \infty$. Moreover, the rate of convergence of this sequence depends only on the modulus of continuity of γ.

We remark that the same result holds in higher dimensions, and its proof is just a simple adaption of the one for dimension 2. For the two-dimensional proof, we are going to do a three-step argument, first recovering the direction of the path, then its signs, only to find the length in the end of the procedure.

6.2 The unsigned directions $\frac{|\Delta_j \gamma_i|}{|\Delta_j \gamma|}$.

For this part we will need some concepts derived directly from the definition of a signature.

Definition 2 (Symmetrized signatures)**.** *Let n, k be two natural numbers. Let also $\mathcal{L}_k^n = \{\ell = (l_1, ..., l_k);\ 0 \leq l_i \leq n,\ \sum_{i=1}^k l_i = n\}$*

- *The multiindex-word set $\mathcal{W}_k^n(w, \ell)$ associated to a word $w = e_{i_1} \cdots e_{i_{k-1}}$ is the set of words $\{w' = w_1 * e_{i_1} * w_2 * \cdots * e_{i_{k-1}} * w_k;\ |w_{i,x}| = 2l_i, |w_{i,y}| = 2n - 2l_i\}$, where $\ell \in \mathcal{L}_k^n$, and $|\tilde{w}_x|$ stands for the quantity of letters x in the word $\tilde{w}$.*

- *The* symmetrized signature *of the word w with respect to n and ℓ is, then, defined as*

$$\mathcal{S}_k^n(w, \ell) = ((2n)!)^k \sum_{w' \in \mathcal{W}_k^n(w,\ell)} C_\gamma(w').$$

Furthermore, we define $\Delta_{k-1} = \{\mathbf{u} = (u_1, ..., u_{k-1}); 0 \leq u_1 \leq u_2 \leq \cdots \leq u_{k-1} \leq 1.\}$, $\Delta_{\mathbf{u},j}\gamma = \gamma(u_{j+1}) - \gamma(u_j)$. With these definitions in hands, a combination of Fubini's theorem and the binomial theorem yields that

$$\sum_{\ell \in \mathcal{L}_k^n} \mathcal{S}_k^n(w, \ell)$$

should be comparable to

$$\int_{\Delta_{k-1}} \prod_{i=1}^{k} |\Delta_{u,j}\gamma|^{2n} \, du,$$

at least for some choice of word w. We let now $\delta(\epsilon) = \sup\{|\gamma'(r) - \gamma'(s)|;\ |r - s| \leq \epsilon\}$ be the modulus of continuity of γ'. The next lemma, which is a simple consequence of the inequality of arithmetic and geometric means and a straightforward majorization argument, allows us to focus on only a restricted region of Δ_{k-1}:

Lemma 3. *Let $k \geq 0$ be such that $\delta(\frac{1}{k}) < \frac{L}{2}$, and $\epsilon_k = \sqrt{2}\left(\sqrt{\frac{\delta(\frac{1}{k})}{L}} + \frac{1}{\sqrt{k}}\right)$. If, for some $j \in [1, k]$, we have $|u_j - \frac{j}{k}| \geq \epsilon_k$, then*

$$\prod_{i=1}^{k} \left(\frac{|\Delta_{u,i}\gamma|}{|\Delta_i\gamma|}\right) < \frac{1}{e}.$$

Finally, using this lemma one can obtain the following precise quantitative comparison stated above:

Theorem 4. *Let $k \geq 0$ such that $\delta(2\epsilon_k) < \frac{L}{6}$. Then there exists a word w^* of length $k-1$ such that, for every $n \in \mathbb{N}$,*

$$\sum_{\ell \in \mathcal{L}_k^n} |\mathcal{S}_k^n(w^*, \ell)| \geq \left(\frac{3}{6^k} - (1 + \frac{3}{6^k})e^{3k \log k - n}\right) L^{k-1} \int_{\Delta_{k-1}} \prod_{i=1}^{k} |\Delta_{u,i}\gamma|^{2n} \, du.$$

With a little aid from this theorem and some work similar to the one needed to prove it, we may already obtain the unsigned directions. We first let $\eta_k = \delta(3\epsilon_k) + \frac{L}{\sqrt{k}}$ and, to shorten notation, $r_j = \frac{|\Delta_j\gamma_1|}{|\Delta_j\gamma|}$. With this definitions, the following holds:

Theorem 5 (Choice of Directions)**.** *Let $k \geq 5$ be such that $\eta_k < \frac{L}{6}$. Then it holds that, for $0 \leq j \leq k$,*

$$\left(\sum_{|w|=k-1} \sum_{\ell;|\frac{l_j}{n}-r_j|\geq\frac{2\eta_k}{L}} |\mathcal{S}_k^n(w,\ell)|\right) / \left(\sum_{|w|=k-1} \sum_{\ell\in\mathcal{L}_k^n} |\mathcal{S}_k^n(w,\ell)|\right) \leq \frac{16}{k^5}\left(\frac{12}{k^4}\right)^k$$

To choose (unsigned) directions from theorem 5 is easy: we simply want to find ρ_j sufficiently close to r_j. To this end, select *any* sequence $\{\rho_j\}_{j=1}^k$ that satisfies

$$\left(\sum_{|w|=k-1} \sum_{\ell;|\frac{l_j}{n}-\rho_j|\leq\frac{2\eta_k}{L}} |\mathcal{S}_k^n(w,\ell)|\right) / \left(\sum_{|w|=k-1} \sum_{\ell\in\mathcal{L}_k^n} |\mathcal{S}_k^n(w,\ell)|\right) > \frac{1}{2}.$$

From theorem 5, such a sequence obviously exists. On the other hand, such a sequence must fulfill $|r_j - \rho_j| < \frac{4\eta_k}{L}$, as otherwise we would fall into the case of theorem 5, a contradiction.

6.3 The signs of directions

Once we have unsigned directions, we must find their signs. The strategy to do so is essentially the same as in the previous section, but with a few new concepts:

Definition 6 (Pseudo-symmetrized signatures)**.** *Let n, k be two natural numbers. Let also $\mathcal{L}_k^n = \{\ell = (l_1, ..., l_k);\ 0 \leq l_i \leq n,\ \sum_{i=1}^k l_i = n\}$*

- *The multiindex-word set $\mathcal{W}_{k,i,x}^n(w,\ell)$ associated to a word $w = e_{i_1}\cdots e_{i_{k-1}}$ is the set of words $\{w' = w_1 * e_{i_1} * w_2 * \cdots * e_{i_{k-1}} * w_k;\ |w_{j,x}| = 2l_i, |w_{j,y}| = 2n - 2l_i,\ \text{if } j \neq i; |w_{i,x}| = 2l_i + 1, |w_{i,y}| = 2n - 2l_i.\}$, where $\ell \in \mathcal{L}_k^n$, and $|\tilde{w}_x|$ stands for the quantity of letters x in the word $\tilde{w}$.*
- *The* pseudo-symmetrized signature *of the word w with respect to n and ℓ is, then, defined as*

$$\mathcal{S}_{k,i,x}^n(w,\ell) = (2n+1)((2n)!)^k \sum_{w'\in\mathcal{W}_{k,i,x}^n(w,\ell)} C_\gamma(w').$$

The definition follows the same spirit for $\mathcal{W}^n_{k,i,y}$ and so on. To go on with the analysis, we choose a word $\tilde{w} = e_{i_1} \cdots e_{i_{k-1}}$ as follows: if $\rho_j > \frac{1}{2}$, we set $e_{i_j} = x$. Otherwise, we set it equal to y. This defined, we pick the signs: we set $a^x_i = +1$ if

$$\frac{\sum_{\ell \in \mathcal{L}^n_k} \mathcal{S}^n_k(\tilde{w}, \ell)}{\sum_{\ell \in \mathcal{L}^n_k} \mathcal{S}^n_{k,i,x}(\tilde{w}, \ell)} > 0,$$

and, on the other case, we set $a^x_i = -1$. The same procedure is made to select a^y_i, and the following theorem shows that, when in the relevant cases, this choice of signs is consistent:

Theorem 7 (Choice of Signs)**.** *Let $k \geq 0$ be sufficiently large, with $\eta_k < \frac{L}{32}$. Then, if $r_i > \frac{2\eta_k}{L}$,*

$$\frac{\sum_{\ell \in \mathcal{L}^n_k} \mathcal{S}^n_k(\tilde{w}, \ell)}{\sum_{\ell \in \mathcal{L}^n_k} \mathcal{S}^n_{k,i,x}(\tilde{w}, \ell)} > \frac{1}{6\epsilon_k L}, \text{ if } \Delta_j \gamma_1 > 0,$$

and

$$\frac{\sum_{\ell \in \mathcal{L}^n_k} \mathcal{S}^n_k(\tilde{w}, \ell)}{\sum_{\ell \in \mathcal{L}^n_k} \mathcal{S}^n_{k,i,x}(\tilde{w}, \ell)} < -\frac{1}{6\epsilon_k L}, \text{ if } \Delta_j \gamma_1 \leq 0.$$

The same holds for γ_2 and a^y_i under the hypothesis of $r_i \leq 1 - \frac{2\eta_k}{L}$.

Now we are ready to approximate the normalized path: let $\theta_j = (a^x_j \rho_j, a^y_j(1-\rho_j))$, and abuse notation to refer to the linear path defined on $[0, \frac{1}{k}]$ and direction θ_j also by θ_j. We finally define

$$\zeta = \frac{1}{k} \theta_1 * \cdots * \theta_k.$$

By the choice of signs – namely, by the fact that a^x_j has the same sign as $\Delta_j \gamma_1$ when $r_j > \frac{2\eta_k}{L}$, and the same for a^y_j when $r_j \leq 1 - \frac{2\eta_k}{L}$ – it is easy to prove that $\frac{\gamma'}{L}$ is $\frac{16\eta_k}{L}$-close to ζ'.

6.4 The length and completion of the procedure

Finally, we approximate the length of the path, with maybe the simplest procedure. We simply pick the shortest word w such that $C_\gamma(w) \neq 0$. Let then m be its length, and define the "reconstructed" length

$$\tilde{L} = \left(\frac{C_\gamma(w)}{C_\zeta(w)} \right)^{1/m}.$$

We have then

Theorem 8 (Approximating lengths)**.** *Let $k \geq 0$ be large enough, such that $\eta_k \leq \frac{(m-1)!}{32L^{m-1}}|C_\gamma(w)|$. Then*

$$|L - \tilde{L}| \leq \frac{32L^m \eta_k}{(m-1)!|C_\gamma(w)|}.$$

The proof is simple: we just use the bounds following from theorem 7 and the explicit definition of signature of a word. With this in hands, we call $\tilde{\gamma} = \tilde{L}\zeta$, and, by theorems 5,7 and 8 we see that

$$\sup_{t\in[0,1]} |\gamma'(t) - \tilde{\gamma}'(t)| \leq 16 \left(1 + \frac{32L^m}{(m-1)!|C_\gamma(w)|}\right)\eta_k.$$

To finish, we remark that our chosen n in the procedure was $n = 4k^2 \log k$, and, as words in $\mathcal{S}^n_{k,i,x}(w,\ell)$ have length $k-1+2n+1+(k-1)2n = 8k^3 \log k + k$, this is the highest in the signature we have to get to achieve our current rate of approximation.

References

[1] Lyons, T. and Xu, W. *Inverting the signature of a path.* eprint arXiv:1406.7833.

[2] Hambly, B. and Lyons, T. *Uniqueness for the signature of a path of bounded variation and the reduced path group.* Ann. of Math. **171** (2010), no. 1, 109–167.

[3] Boedihardjo, B., Geng, X., Lyons, T. and Yang, D. *The signature of a rough path: uniqueness.* Adv. Math. **293** (2016), 720–737.

João Pedro Ramos, University of Bonn
email: **jpgramos@math.uni-bonn.de**

7 Characteristic functions of measures on geometric rough paths

after I. Chevyrev and T. Lyons [1]
A summary written by Michał Warchalski

Abstract

We define a characteristic function for probability measures on the signatures of geometric rough paths and study its properties. We further apply it to study uniqueness of random variables based on their expected signature.

7.1 Overview

We treat rough paths in the sense of Lyons [2]. Let us quickly recall some necessary definitions. Further our setting is more general with a (possibly infinite dimensional) Banach space V, however initially one can think of $V = \mathbb{R}^d$, since ultimately the results are applied to this case.

Let $p \geq 1$, $T > 0$ and

$$\Delta_{[0,1]} = \{(s,t)\colon 0 \leq s \leq t \leq T\}$$

Rough paths are functions taking values in the (truncated) tensor algebra, which is defined for $n \geq 0$ as

$$T^n(V) = \bigoplus_{0 \leq k \leq n} V^{\otimes k} = V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus .. \oplus V^{\otimes n}$$

The space of p-rough paths $\Omega_p(V)$ is the collection of all continuous maps $\mathbf{x}\colon \Delta_{[0,1]} \to T^{\lfloor p \rfloor}$ satisfying

- (multiplicativity) $\mathbf{x}_{s,t}\mathbf{x}_{t,u} = \mathbf{x}_{s,u}$ for all $0 \leq s \leq t \leq u \leq T$,
- (controlled p-variation) for some control function ω

$$\sup_{0 \leq k \leq \lfloor p \rfloor} \left((k/p)!\beta_p \|\mathbf{x}^k_{s,t}\|\right)^{p/k} \leq \omega(s,t)$$

for all $(s,t) \in \Delta_{[0,T]}$. The point of the control is to ensure that the p-variation

$$\|\mathbf{x}\|_{p-var;[0,T]} := \sum_{0\le k\le\lfloor p\rfloor} \sup_{\mathcal{D}\subset[0,T]} \left(\sum_{t(j)\in\mathcal{D}} \left((k/p)!\beta_p \|\mathbf{x}^k_{t(j),t(j+1)}\|\right)^{p/k} \right)^{1/p}$$

is finite.

We say that $\mathbf{x}(n) \in \Omega_p$ converges to $\mathbf{x} \in \Omega_p$ in the p-variation topology if

- the p-variations of $\mathbf{x}$ and $\mathbf{x}(n)$ for $n \ge 1$ are controlled
- for a sequence $a_n \to \infty$, $\sup_{0\le k\le\lfloor p\rfloor} \left((k/p)!\beta_p a_n \|\mathbf{x}^k_{s,t}\|\right)^{p/k} \le \omega(s,t)$ for all $(s,t) \in \Delta_{[0,T]}$.

A fundamental result result in the theory of rough paths is that any path $x \in \Omega_p$ can be uniquely lifted to $S(\mathbf{x})\colon \Delta_{[0,T]} \to \prod_{k\ge0} V^{\otimes k}$. In other words, the more rough a path (the bigger p) is the more information (the more initial coordinates) we need to uniquely determine it. Precisely, we have the following result: let $\Omega E_p := S(\Omega_p)$ together with the p-variation topology, with $\sup_{0\le k\le\lfloor p\rfloor}$ replaced by $\sup_{0\le k}$; then S is a homeomorphism between Ω_p and ΩE_p.

We shall be working with the space of geometric rough paths. Consider a rough path $\mathbf{x} \in \Omega_p$ and its lift $S(\mathbf{x}) \in \Omega E_p$. Define the set of geometric rough paths as the closure of "regular" paths Ω_1 inside Ω_p.

Definition 1. *The space of geometric rough paths $G\Omega_p$ is the closure of Ω_1 in Ω_p.*

For $p \ge 1$, set $S_p(V) = \{S(\mathbf{x})_{0,T} \colon x \in G\Omega_p\} \subset E$ as the set of signatures of geometric p-rough paths. It follows from the above discussion that S_1 is dense in S_p.

7.1.1 Expected signature

The value $S(\mathbf{x})_{0,T}$ is called *the signature* of $\mathbf{x}$. It is morally "the sequence iterated integrals of the path $x_{0,\cdot}\colon [0,T] \to V$" and this relation is actually precise in the sense of Young for exponents in the range $1 \le p < 2$. The signature $(S(\mathbf{x})^0_{0,T}, S(\mathbf{x})^1_{0,T}, S(\mathbf{x})^2_{0,T}, ...)$ is thus an element of $\prod_{k\ge0} V^{\otimes k}$

and the expected signature is defined as the sequence of expectations taken coordinatewise

$$\text{ExpSig}(S(\mathbf{x})_{0,T}) := (\mathbb{E}[S(\mathbf{x})^0_{0,T}], \mathbb{E}[S(\mathbf{x})^1_{0,T}], \mathbb{E}[S(\mathbf{x})^2_{0,T}], ...)$$

One can show that if the expected signature decays fast, then $S(\mathbf{x})$ is *weakly integrable*. Moreover, one can prove that in such situation the expected signature is *unique* in distribution:

Proposition 2. *For two rough paths* $\mathbf{x}$*,* $\mathbf{y}$ *if*

$$\mathit{ExpSig}(S(\mathbf{x})_{0,T}) = \mathit{ExpSig}(S(\mathbf{y})_{0,T}),$$

and $\mathit{ExpSig}(S(\mathbf{x})_{0,T})$ *decays fast enough, then* $S(\mathbf{x})_{0,T} = S(\mathbf{y})_{0,T}$ *in distribution.*

7.1.2 Characteristic functions of rough paths

Recall, the characteristic function of a random variable X is simply its Fourier transform given by $E[e^{i\lambda X}]$. Since we further treat rough paths as paths taking values in a topological group we follow the classical extension of the Fourier transform: we define the characteristic function as

$$\phi_X(M) = \mathbb{E}[M(X)],$$

where X is a random signature and M is a unitary representation of the group. One of the main results is that the characteristic function uniquely determines a rough path.

Proposition 3. *For two rough paths* $\mathbf{x}, \mathbf{y}$*:* $S(\mathbf{x})_{0,T} = S(\mathbf{y})_{0,T}$ *in distribution if and only if* $\phi_{S(\mathbf{x})_{0,T}} = \phi_{S(\mathbf{y})_{0,T}}$*.*

7.2 General setting

Most of the arguments are worked out in the following setting.

Let V be a Banach space and $T(V) = \bigoplus_{k\geq 0} V^{\otimes k}$ be the tensor algebra of V. The multiplication is given by the tensor product and extended linearly to the whole space.

If γ and ξ are semi-norms on V and W, then by $\gamma \otimes \xi$ we denote *the projective semi-norm* on $V \otimes W$:

$$\gamma \otimes \xi(x) := \inf \left\{ \sum_{i=1}^{n} \gamma(a_i)\xi(b_i) \colon x = \sum_{i=1}^{n} a_i b_i \right\}$$

Denote by $V \hat{\otimes} W$ the completion of $V \otimes W$ in this norm.

In this manner we define *the projective extension* of a semi-norm γ on V as the semi-norm

$$\exp(\gamma) := \sum_{k \geq 0} \gamma^{\otimes k}.$$

We call a family of semi-norms Λ on V *fundamental* if for every semi-norm ξ on V, there exists $\gamma \in \Lambda$ such that $\gamma \geq \xi$. Now, we can finally define what we meant by *fast decaying* elements of $P := \prod_{k \geq 0} V^{\hat{\otimes} k}$. Let Λ be a fundamental family of semi-norms on V, define

$$E = \{x \in P \colon \forall \gamma \in \Lambda,\ \sum_{k \geq 0} \gamma^{\otimes k} < \infty\}.$$

In particular the signature $S(\mathbf{x})_{0,T}$ of a rough path lies in E.

$T(V)$ is a *Hopf algebra* with

- coproduct: $\Delta v = 1 \otimes v + v \otimes 1$ for $v \in V$, extended by $\Delta(v_1 ... v_k) = \Delta(v_1)...\Delta(v_k)$ for $v_1 ... v_k \in V^{\otimes}$,
- antipode: $\alpha(v_1 ... v_k) = (-1)^k v_k ... v_1$ for $v_1 ... v_k \in V^{\otimes k}$.

We denote by

$$G(V) = \{g \in E \colon \Delta(g) = g \otimes g,\ g \neq 0\}.$$

the group of *group-like elements* of E, their inverse is given by the antipode.

7.3 Some results

Chen [3] proved that the signature of a path in $S_1(\mathbb{R}^d)$ is a group-like element of $E(\mathbb{R}^d)$, what immediately implies the same statement for paths in $S_p(\mathbb{R}^d)$. This lets us apply results in the general setting to $V = \mathbb{R}^d$. Furthermore, we have the following.

Proposition 4. *Let $p \geq 1$. Then the set of signatures of gemetric p-rough paths $S_p(\mathbb{R}^d)$ is a Borel subset of $G(\mathbb{R}^d)$.*

Let $\mathcal{A}(V)$ be the family of finite dimensional representations of E coming from extensions of linear maps from V to the Lie algebra of anti-Hermitian operators on a finite dimensional Hilbert space H. Note that $M(\alpha x) = M(x)^*$ for $x \in E$, which implies that $M(x)$ is unitary for all $x \in G$. Hence, every $M \in \mathcal{A}$ is a unitary representation of G. The Stone-Weierstrass theorem implies that if a $*$-subalgebra $\mathcal{B} \subset C_b(G, \mathbb{C})$ separates points of G, then two (tight) measures $\mu = \nu$ if and only if $\mu(M) = \nu(M)$ for all $M \in \mathcal{B}$.

One can actually prove that the algebra $\mathcal{A}(\mathbb{R}^d)$ separates points of $E(\mathbb{R}^d)$, which lets us conclude

Proposition 5. *For Borel probability measures μ and ν on $G(\mathbb{R}^d)$, it holds that $\mu = \nu$ if and only if $\mu(M) = \nu(M)$ for all $M \in \mathcal{A}(\mathbb{R}^d)$.*

It follows from the above proposition that if X is a $G(\mathbb{R}^d)$-valued random variable and μ is its distribution, then the function $\phi_X := \mu|_{\mathcal{A}}$ uniquely characterizes μ and thus implies Proposition 3.

Since for ExpSig(X) $\in E$ and $f \in E'$, $E[f(X)]$ is determined by ExpSig(X), we obtain from the previous proposition:

Proposition 6. *Let X, Y be $G(\mathbb{R}^d)$-valued random variables. If ExpSig$(X) =$ ExpSig(Y) and ExpSig$(X) \in E$, then $X = Y$ in distribution.*

Recall, the signature of a rough path is $G(\mathbb{R}^d)$-valued - in view of this fact Proposition 2 follows from the above.

References

[1] Ilya Chevyrev, Terry Lyons *Characteristic functions of measures on geometric rough paths.* arXiv:1307.3580

[2] Terry Lyons, Zhongmin Qian *System control and rough paths.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2002

[3] Kuo-Tsai Chen *Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula.* Ann. of Math. (2), 65:163-178, 1957.

Michał Warchalski, University of Bonn
email: **mwarchal@math.uni-bonn.de**

8 Decay rate of iterated integrals of branched rough paths

after H. Boedihardjo [1]
A summary written by Marco Fraccaroli

Abstract

We prove a factorial decay estimate, conjectured by Gubinelli [2], for the iterated integrals of non-geometric rough paths. We also explain through a counterexample why the conventional approach of using the neoclassical inequality fails. The proof of our theorem involves instead a concavity estimate for sums over rooted trees and a non-trivial extension of Lyons' proof for the factorial decay of iterated Young's integrals [3].

8.1 Introduction

The iterated integrals of a path x, defined by

$$X_{0,1}^n := \int_{0<s_1<\ldots<s_n} dx_{s_1} \otimes \ldots \otimes dx_{s_n}$$

as n varies, arise naturally from the Taylor's expansion of a controlled differential equation driven by the path and play a fundamental role in the theory of rough paths. The convergence of the series expansion is often studied using their decay. Lyons in [4] proved a factorial decay for geometric rough paths case through a highly non-trivial binomial-type inequality known as the neoclassical inequality.

Gubinelli in [2] proposed a theory of *non-geometric* rough paths, known as the branched rough paths, and conjectured that an analogous tree factorial decay holds even in this case.

Therefore the main result we want to prove is the following:

Theorem 1. *Let $0 < \gamma \leq 1$ and $N = \lfloor \gamma^{-1} \rfloor$. Let X be a γ-branched rough path. For all rooted trees τ and all $s \leq t$,*

$$|\langle X_{s,t}, \tau \rangle \leq \frac{\bar{c}_N^{|\tau|}(t-s)^{\gamma|\tau|}}{\tau!^\gamma}$$

where

$$\bar{c}_N = 6\exp\left(7\sum_{i=0}^{N+1}(N+1)^{i+1}\right)|\mathcal{T}^{\mathcal{N}}|^{2-2\gamma}2^{(N+1)\gamma}\zeta((N+1)\gamma)N!^{\gamma}\max_{1\leq|\sigma|\leq N}\|X\|_{\gamma,\sigma}^{|\sigma|^{-1}}$$

and $\mathcal{T}^N$ *is the set of unlabelled rooted trees with at most* N *vertices.*

Remark 2. *Unless explicitly stated, from now on* $0 < \gamma \leq 1$ *and* $N = \lfloor \gamma^{-1} \rfloor$.

The theorem, together with the Hairer-Kelly way of extending a geometric rough path to a branched one, gives another proof for the factorial decay for the former ones and in some cases it even provides a sharper estimate.
The strategy of proof cannot rely on the neoclassical inequality as in the case of γ-geometric rough paths since we can provide a counterexample for a version of it for rooted trees.
Instead we want to follow Lyons' approach in [3], which proved the factorial decay for γ-geometric rough paths in the $\gamma > \frac{1}{2}$ case without neoclassical inequality. We define

$$X^n_{u,t} = \lim_{\substack{\mathcal{P}\subset[u,t]\\|\mathcal{P}|\to 0}}\sum_{i=1}^{m-1}\sum_{k=1}^{N} X^{n+1-k}_{s,t_i}\star X^k_{t_i,t_{i+1}}$$

where $N = \lfloor \gamma^{-1} \rfloor$, $n \geq N+1$ and the limit is taken as the mesh size $\max_{0\leq i<m}|t_{i+1}-t_i|$ of the partition $\mathcal{P} = \{s = t_0 < t_1 < \ldots < t_m = t\}$ goes to zero. The purpose is to use Young's method of estimating $X^n_{u,t}$ by successively removing partition points from the partition. In order to do this we need to find a control function $(s,t) \to R_u(s,t)$ (i.e. such that $R_u(s,v)+R_u(v,t) \leq R_u(s,t)$) which dominates the function $(s,t) \to \omega_u(s,t)$ defined by

$$\omega_u(s,t) := \|\sum_{k\geq N+1} X^{n-k}_{u,s}\star X^k_{s,t}\|_{\mathcal{T},\gamma,\beta}$$

and satisfies some binomial properties. The strategy of the proof consists of:

1. prove a bound for the multiplication operator $\star$ with respect to some norm, analogously to the bound of tensor product $\|a\otimes b\| \leq \|a\|\|b\|$ for $a \in V^{\otimes m}$ and $b \in V^{\otimes n}$ in the geometric case;

2. prove that our function R is compatible with the tree multiplication, which corresponds to the multiplication of the coordinate components of the path in the geometric case;

3. prove that our function R is compatible with the operation of joining trees to a single root, which corresponds to integrating against the path in the geometric case.

8.2 Notation and terminology for branched rough paths

A *rooted tree* is a connected, rooted graph such that, for every vertex in the graph, there exists a unique path from the root to the vertex. Let 1 denote the empty tree and $\bullet$ the one consisted of a single vertex. A *forest* is a finite set of rooted trees. We use bold symbols (e.g. $\boldsymbol{\tau}$) for forests and normal symbols (e.g. τ) for rooted trees.
We denote with $\mathcal{T}$ the set of rooted trees and with $\mathcal{F}$ the one of forests.
We define a commutative multiplication $\cdot$ on $\mathcal{F}$ by $x \cdot y := x \cup y$.
For $\sigma = \{\tau_1, \ldots, \tau_n\} = \tau_1 \ldots \tau_n \in \mathcal{F}$, where $\tau_1, \ldots, \tau_n$ are rooted non-empty trees, let $[\sigma]_\bullet$ denote the rooted tree obtained by joining the roots of $\tau_1, \ldots, \tau_n$ to the vertex $\bullet$.
We denote with $\mathcal{H}$ the formal vector space spanned by $\mathcal{F}$ over $\mathbb{R}$.
For a forest τ let $c(\tau)$ denote the number of non-empty trees in τ and $|\tau|$ the total number of vertices in the forest. For each tree τ the tree factorial $\tau!$ is defined inductively by

$$
\begin{aligned}
\bullet! &= 1, \\
[\tau_1, \ldots, \tau_n]_\bullet! &= |[\tau_1, \ldots, \tau_n]_\bullet| \tau_1! \ldots \tau_n!, \\
\tau_1 \ldots \tau_n! &= \tau_1! \ldots \tau_n!.
\end{aligned}
$$

A coproduct of rooted trees can be inductively defined as $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ by

$$
\begin{aligned}
\Delta 1 &= 1 \otimes 1, \\
\Delta[\tau_1, \ldots, \tau_n]_\bullet &= [\tau_1, \ldots, \tau_n]_\bullet \otimes 1 + \sum \tau_{\mathbf{1}}^{(\mathbf{1})} \ldots \tau_{\mathbf{n}}^{(\mathbf{1})} \otimes [\tau_1^{(2)}, \ldots, \tau_n^{(2)}]_\bullet, \\
\Delta(\tau_1 \ldots \tau_n) &= \Delta\tau_1 \ldots \Delta\tau_n.
\end{aligned}
$$

where the sum in the second definition is taken over all terms $\tau_{\mathbf{i}}^{(\mathbf{1})}$ and $\tau_i^{(2)}$ in $\Delta\tau_i = \sum \tau_{\mathbf{i}}^{(\mathbf{1})} \otimes \tau_i^{(2)}$.
The coproduct Δ also has an interpretation in terms of admissible cuts. An admissible cut c of a rooted tree τ is a set of edges of the rooted tree such that, for any vertex in τ, the path from the root to the vertex passes through

at most one element in the cut. If we denote by $\tau_{\mathbf{c}}^{(\mathbf{1})}$ and $\tau_c^{(2)}$ respectively the components in $\tau \setminus c$ that is disconnected from the root and the component that is connected to the root, then

$$\Delta\tau = \tau \otimes 1 + \sum_{c \text{ admissible cut}} \tau_{\mathbf{c}}^{(\mathbf{1})} \otimes \tau_c^{(2)}.$$

8.3 Counterexample to the tree neoclassical inequality

The following lemma exhibits the counterexample for a weaker version of the neoclassical inequality which would have been sufficient in proving the factorial decay for the iterated integrals of branched rough paths (as Gubinelli showed in [2]).

Lemma 3. *Let τ_n be the tree $[\bullet^n]_\bullet$. Then for all $0 \le \gamma < 1$, for all $\beta > 0$, there exist $a, b > 0$ such that, as $n \to \infty$,*

$$(a+b)^{-\gamma|\tau_n|} \sum \left(\frac{\tau_n!}{\tau_{\mathbf{n}}^{(\mathbf{1})}!\tau_n^{(2)}!} \right)^{\gamma} \frac{1}{\beta^{c(\tau_{\mathbf{n}}^{(\mathbf{1})})+c(\tau_n^{(2)})}} a^{\gamma|\tau_{\mathbf{n}}^{(\mathbf{1})}|} b^{\gamma|\tau_n^{(2)}|} \to \infty.$$

8.4 Tools for the proof

8.4.1 Bound for the multiplication operator $\star$

Let $X \in \mathcal{H}^*$. Define a linear functional $X^k \in \mathcal{H}^*$ for every $\tau \in \mathcal{F}$ by

$$\langle X^k, \tau \rangle := \begin{cases} \langle X, \tau \rangle & \text{if } |\tau| = k, \\ 0 & \text{otherwise.} \end{cases}.$$

Then for $X, Y \in \mathcal{H}^*$ we define the linear functional $X^n \star Y^k \in \mathcal{H}^*$ by

$$\langle X^n \star Y^k, \tau \rangle := \langle X^n \otimes Y^k, \Delta\tau \rangle$$

for every $\tau \in \mathcal{F}$. Let

$$\|X^k\|_{\mathcal{T},\gamma,\beta} := \max_{\tau \in \mathcal{T}, |\tau| = k} |\langle X, \tau \rangle| \frac{\beta^{c(\tau)} \tau!^\gamma}{|\tau|!^\gamma},$$

and

$$\|X^k\|_{\mathcal{F},\gamma,\beta} := \max_{\tau \in \mathcal{F}, |\tau| = k} |\langle X, \tau \rangle| \frac{\beta^{c(\tau)} \tau!^\gamma}{|\tau|!^\gamma}.$$

Lemma 4. *Let*

$$c_k := \exp\left[\sum_{i=1}^{k} k^i(1-\gamma)\right], \beta \geq c_k.$$

Let $X, Y \in \mathcal{H}^*$. Then for $n \geq 1$,

$$\|X^n \star Y^k\|_{\mathcal{T},\gamma,\beta} \leq c_k |\mathcal{T}^k|^{1-\gamma} \beta^{-1} \|X^n\|_{\mathcal{F},\gamma,\beta} \|Y^k\|_{\mathcal{T},\gamma,\beta},$$

where $\mathcal{T}^k$ denotes the set of rooted trees with k vertices.

Remark 5. *Notice that the norm of the multiplication operation can be bounded by a function of k, independently of n.*

The proof requires the following concavity estimate, which in the non-geometric case has the same role of the neoclassical inequality in the geometric one, namely of taking the fractional power γ outside a sum.

Lemma 6. *For any rooted tree σ, let $c_{|\sigma|}$ be defined as above and $\beta \geq c_{|\sigma|}$. For all rooted trees τ such that $\sigma \subsetneq \tau$, we have*

$$\sum_{\tau^{(2)}=\sigma} \frac{\beta^{-c(\tau^{(\mathbf{1})})}}{\tau^{(\mathbf{1})}!^{\gamma}} \leq c_{|\sigma|} \beta^{-1} \left(\sum_{\tau^{(2)}=\sigma} \frac{1}{\tau^{(\mathbf{1})}!} \right)^{\gamma}.$$

Remark 7. *The key point of this lemma is that the constant we lose by taking the power γ outside the sum, $c_{|\sigma|}$, depends only on $|\sigma|$ but not on $|\tau|$.*

8.4.2 Compatibility of our estimate with tree multiplication

For each $a, b > 0$ define a one-dimensional path ρ_a^b by

$$\rho_a^b(t) := \frac{1}{b}(t-a)^b.$$

Let $\Delta_m(r, r')$ denote the m-dimensional simplex

$$\{(s_1, \ldots, s_m) \in \mathbb{R}^m : r < s_1 < \ldots < s_m < r'\}.$$

For a one-dimensional path ρ, we define

$$S^m(\rho)_{s,t} := \int_{\Delta_m(r,r')} d\rho(s_1) \ldots d\rho(s_m).$$

Lemma 8. *Let X be a γ-branched rough path. Let*

$$\hat{c}_N := 3|\mathcal{T}^N|^{1-\gamma}(N+1)^{3(1-\gamma)}\exp\left[2(N+1)\right], \beta \geq \hat{c}_N.$$

Suppose that for all $n \leq M$ and $u \leq s \leq t$,

$$\|\sum_{k\geq N+1} X_{u,s}^{n-k} \star X_{s,t}^{k}\|_{\mathcal{T},\gamma,\beta} \leq \left[\frac{1}{(n-N-1)!} S^{(N+1)}\left(\rho_u^{\frac{n}{N+1}}\right)_{s,t}\right]^{\gamma}.$$

Then for all $n \leq M$ and $u \leq s \leq t$,

$$\|\sum_{k\geq N+1} X_{u,s}^{n-k} \star X_{s,t}^{k}\|_{\mathcal{F},\gamma,\beta\hat{c}_N^{-1}} \leq \left[\frac{1}{(n-N-1)!} S^{(N+1)}\left(\rho_u^{\frac{n}{N+1}}\right)_{s,t}\right]^{\gamma}.$$

8.4.3 Estimate for the remainder of a coproduct sum of branched rough paths

Lemma 9. *Let X be a γ-branched rough path. If for any $0 \leq n \leq N$,*

$$\|X_{s,t}^{n}\|_{\mathcal{T},\gamma,\beta} \leq \left[\frac{(t-s)^n}{n!}\right]^{\gamma},$$

and

$$\beta \geq 6\exp\left(7\sum_{i=1}^{N+1}(N+1)^i\right)\sum_{r=2}^{\infty}\left(\frac{2}{r-1}\wedge 1\right)^{(N+1)\gamma}|\mathcal{T}^N|^{1-\gamma}$$

then for any $n \geq N+1$,

$$\|\sum_{k\geq N+1} X_{u,s}^{n-k} \star X_{s,t}^{k}\|_{\mathcal{T},\gamma,\beta} \leq \left[\frac{1}{(n-N-1)!} S^{(N+1)}\left(\rho_u^{\frac{n}{N+1}}\right)_{s,t}\right]^{\gamma}.$$

References

[1] H. Boedihardjo, *Decay rate of iterated integrals of branched rough paths.* Preprint, arXiv: 1501.05641v3 (2015).

[2] M. Gubinelli, *Ramification of rough paths.* Journal of Differential Equations 248, 693-721 (2010).

[3] T. Lyons, *Differential equations driven by rough signals (I): an extension of an inequality of L.C. Young.* Mathematical Research Letters 1, 451-464 (1994).

[4] T. Lyons, *Differential equations driven by rough signals.* Revista Matematica Iberoamericana 14 (2), 215-310 (1998).

Marco Fraccaroli, University of Bonn
email: **s6mafrac@uni-bonn.de**

9 Physical Brownian Motion in Magnetic Field as Rough Path

after P. Friz, P. Gassiat, and T. Lyons [1]
A summary written by Kevin O'Neill

Abstract

We summarize [1], which shows that under the influence of a magnetic field, physical Brownian motion (meaning with nonzero mass) converges in the sense of rough paths to the massless case plus a nontrivial area term. In this summary, we also give brief introductions to the tools needed to understand this main theorem.

9.1 Introduction

A Brownian motion is a stochastic process used to model the random motions of a massless particle. Mathematically speaking, we have the following definition [3]:

Definition 1. *A real-valued stochastic process W_t is a one-dimensional Brownian motion if*
(i) $W_0(\omega) = 0$ for all ω.
(ii) The map $t \mapsto W_t(\omega)$ is a continuous function for all ω, and
(iii) For every $t, h \geq 0, W_{t,t+h} := W_{t+h} - W_t$ is independent of $(W_u : 0 \leq u \leq t)$ and has a Gaussian distribution of mean 0 and variance h.
A d-dimensional Brownian motion is an R^d-valued stochastic process with each component an independent one-dimensional Brownian motion.

This is intended to describe phenomena such as a speck of dust floating in a glass of water, however in the real world no speck of dust or other particle has precisely zero mass. For this reason, we consider a *physical* Brownian motion described by the stochastic differential equation

$$m\ddot{x} = -A\dot{x} + \xi \tag{1}$$

and take the limit of solutions as $m \to 0$. In (1), m is the mass of the particle, ξ is the (distributional) derivative of a Brownian motion W, and A is a $d \times d$ diagonal matrix with positive diagonal entries. If our particle is charged and moving through a constant magnetic field, one may show that the position $X(t)$ of our particle obeys the system

$$dX = \frac{1}{m} Y\, dt \tag{2}$$

$$dY = -\frac{1}{m} MY\, dt + dW \tag{3}$$

where M is a $d \times d$ matrix whose eigenvalues all have strcitly positive real part. Our goal is to examine the limit of solutions to (2) and (3) as $m \to 0$.

9.2 Rough Paths

It may be shown that a Brownian path (i.e., a realization of Brownian motion) is almost surely α-Hölder continuous for any $\alpha < 1/2$, but not for $\alpha = 1/2$. This presents a challenge in solving any stochastic differential equations such as (2) and (3); in particular, one may show that the solution is not continuous as a function of the driving noise. However, this issue may be resolved by the introduction of rough paths, as the solution map with domain the space of rough paths may be continuous.[2]

Definition 2. *A rough path* $\mathbf{X}$ *in* $\mathbb{R}^d$ *on an interval* $[0, T]$ *is a continuous function* $X : [0, T] \to \mathbb{R}^d$ *equipped with a continuous function* $\mathbb{X} : [0, T]^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$ *satisfying the relation*

$$\mathbb{X}_{s,t} - \mathbb{X}_{u,t} - \mathbb{X}_{s,u} = X_{s,u} \otimes X_{u,t}. \tag{4}$$

The α*-Hölder rough path metric is given by*

$$\rho_\alpha(\mathbf{X}, \mathbf{Y}) := \sup_{s \neq t \in [0,T]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^\alpha} + \sup_{s \neq t \in [0,T]} \frac{|\mathbb{X}_{s,t} - \mathbb{Y}_{s,t}|}{|t - s|^{2\alpha}}.$$

9.3 Stochastic Integration

As long as $\alpha \geq 1/2$, there is a canonical choice of $\mathbb{X}$, given by

$$\mathbb{X}_{s,t} = \int_s^t X_{s,r} \otimes dX_r,$$

the Young integral which shows up in a second-order Euler process. Since Brownian motion is only α-Hölder continuous almost surely for $a < 1/2$, there is no such canonical choice. While there are an infinite number of choices of $\mathbb{X}$ for any given X under which (4) is satisfied, we restrict our attention to two choices which are based on two different forms of stochastic integration.

The first of these is Itô integration, which is defined as

$$\mathbb{W}^{\text{Itô}}_{0,t} = \int_0^t W_{0,r} \otimes dW_r = \lim_{n\to\infty} \sum_{[t_{i-1},t_i]\in D_n} W_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}),$$

where D_n is a sequence of partitions with mesh going to 0. One may show this limit converges in probability. The other kind of stochastic integration is called Stratonovich integration and may be defined as

$$\mathbb{W}^{\text{Strat}}_{0,t} = \int_0^t W_{0,r} \otimes \circ dW_r = \int_0^t W_{0,r} \otimes dW_r + \frac{1}{2}[W,W]_t$$

as in [2]. However, this just reduces to $\mathbb{W}^{\text{Strat}}_{0,t} = \mathbb{W}^{\text{Itô}}_{0,t} + \frac{1}{2}I(t-s)$. Both of these are valid ways to turn a Brownian path into a rough path (with probability 1), though we briefly note that Stratonovich integration gives what is known as a geometric rough path while Itô integration does not.

9.4 Theorem Statement and Remarks

Now that we have a notion of Brownian motion which includes second-order terms, we would like to have some statement which says that physical Brownian motion converges to the usual Brownian motion under the metric ρ_α. However, it turns out that in our particular scenario, we have an extra non-trivial second-order term in the limit. Additionally, we must recall that these are stochastics processes, so we may not simply use ρ_α without consideration of the underlying probability measure. Taking thiese factors into account, the main theorem proved in [1] is as follows:

Theorem 3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be solutions to (2) and (3). Let* $\hat{\mathbf{W}} = (W, \hat{\mathbb{W}})$*, where*

$$\hat{\mathbb{W}}_{s,t} = \mathbb{W}_{s,t} + (t-s)\frac{1}{2}(MC - CM^*)$$

and $C = \int_0^\infty e^{-Ms}e^{-M^*s}ds$*. Then, as* $m \to 0$*,* MX *converges to* $\hat{\mathbf{W}}$ *in* L^q *and under the metric* ρ_α *for any* $q \geq 1$ *and* $\alpha \in (1/3, 1/2)$*.*

Remark 4. *In the case where there is no magnetic field, M is symmetric and $\hat{\mathbf{W}} = (W, \mathbb{W})$ so there is no extra (area) term in the above limit.*

Remark 5. *If one instead chooses to consider the differential equation*

$$m\ddot{x} = -M\dot{x} + \dot{\gamma},$$

where γ is α-Hölder continuous of order $\alpha > 1/2$, then there is no area term in the limit.

To summarize the above remarks, the major hypotheses are necessary for the conclusion.

9.5 Idea of Proof

The idea behind the proof is as follows. Begin by rescaling with $m = \epsilon^2$ and denote solutions as X^ϵ and Y^ϵ. Defining $\tilde{W}_t = \epsilon W_{\epsilon^{-2}t}$, it suffices to consider the system

$$d\tilde{Y} = -M\tilde{Y}dt + d\tilde{W} \qquad d\tilde{X} = \tilde{Y}dt,$$

since under identical initial data, $(Y_t^\epsilon, \epsilon^{-1}X_t^\epsilon) = (\tilde{Y}_{\epsilon^{-2}t}, \tilde{X}_{\epsilon^{-2}t})$. Convergence of the first-order terms follows from the equation $MX_t^\epsilon = W_t - \epsilon Y_{0,t}^\epsilon$ and the fact that $\epsilon\tilde{Y}_{\epsilon^{-2}t} = \epsilon Y_t^\epsilon \to 0$ in L^2 uniformly in t.

Next, one may show that the matrix C is equal to $\mathbf{E}(\tilde{Y}_0^{\text{stat}} \otimes \tilde{Y}_0^{\text{stat}})$, and furthermore that

$$\int_0^t Y_s^\epsilon \otimes Y_s^\epsilon ds \to tC$$

as $\epsilon \to 0$, using either the ergodic theorem, or direct computation (the latter of which may determine rates of convegence). This allows one to compute the limit of $\int_0^t MX_s^\epsilon \otimes d(MX^\epsilon)_s$, getting $\mathbb{W}_{0,t} + t(MC - \frac{1}{2}I)$. This gives the desired limit since $(MC - \frac{1}{2}I)$ is antisymmetric.

However, we have only shown pointwise convergence (with respect to the probability measure). In order to establish convergence with respect to ρ_α and in L^q, we prove uniform rough path bounds in L^q, meaning that

$$\sup_{0<\epsilon\leq 1} \mathbf{E}[||MX^\epsilon||_\alpha^q] < \infty, \qquad \sup_{0<\epsilon\leq 1} \mathbf{E}\left[\left|\left|\int MX^\epsilon \otimes d(MX^\epsilon)\right|\right|_{2\alpha}^q\right] < \infty.$$

This may be proven via use of the Kolmogorov criterion for rough paths (also used to establish 2α-Hölder regularity of area terms of Brownian motion), which states we need only show

$$\sup_{0<\epsilon\leq 1} \mathbf{E}[|X^\epsilon_{s,t}|^q] \lesssim |t-s|^{q/2} \qquad \sup_{0<\epsilon\leq 1} \mathbf{E}\left[\left|\int_s^t X^\epsilon_s \otimes dX^\epsilon\right|^q\right] \lesssim |t-s|^q.$$

It should be noted that important elements of the remainder of the proof are 1) interpolation of convergence of rough paths via these uniform bounds, and 2) the use of the positive real parts of the eigenvalues of M to establish finiteness of certain integrals.

References

[1] P. Friz, P. Gassiat, T. Lyons, *Physical Brownian Motion in Magnetic Field as Rough Path.* Trans. Amer. Math. Soc. **367** (2015), no. 11, 7939-7955;

[2] P. Friz and M. Hairer, *A Course on Rough Paths. With an Introduction to Regularity Structures* Springer International Publishing, Switzerland, 2014.

[3] P. Friz and N. Victoir, *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications* Cambridge University Press, Cambridge, 2010.

Kevin O'Neill, University of California, Berkeley
email: **oneill@math.berkeley.edu**

10 Reflected rough differential equations

after S. Aida [1]
A summary written by Joris Roos

Abstract

The author of the paper proves existence of solutions for reflected differential equations on a domain in $\mathbb{R}^d$ with the boundary satisfying certain conditions. In particular, the boundary is not necessarily smooth. The driving paths are assumed to be continuous and of finite p-variation with $1 \leq p < 2$, or p-rough with $2 \leq p < 3$.

10.1 The Skorohod problem

Let D be a connected subset of $\mathbb{R}^d$. We will need certain conditions on the boundary of D which are formulated in terms of inward unit normal vectors. The set of *inward unit normal vectors* at a point $x \in \partial D$ is defined by

$$\mathcal{N}_x = \{\mathbf{n} \in S^{d-1} \,:\, \exists r > 0 \,\text{s.t.}\, \mathbf{n} \in \mathcal{N}_{x,r}\},$$

where

$$\mathcal{N}_{x,r} = \{\mathbf{n} \in S^{d-1} \,:\, B(x - r\mathbf{n}, r) \cap D = \emptyset\}.$$

Here $B(x,r) = \{y \in \mathbb{R}^d \,:\, |x-y| < r\}$ denotes the open ball of radius $r > 0$ around a point $x \in \mathbb{R}^d$. Clearly we have $\mathcal{N}_{x,r} \subseteq \mathcal{N}_{x,s}$ if $0 < s \leq r$.
For a path $(x_t)_{t\in[0,T]}$ and $0 \leq s \leq t \leq T$, $1 \leq p < \infty$ we set

$$\|x\|_{p;[s,t]} := \sup_{N\in\mathbb{N}, s=t_0<t_1<\cdots<t_N\leq t} \left(\sum_{i=0}^{N-1} |x_{t_{i+1}} - x_{t_i}|^p \right)^{1/p},$$

$$\|x\|_{\infty;[s,t]} := \max_{s\leq u\leq v\leq t} |x_u - x_v|.$$

Recall that we say x_t has *finite p-variation* if $\|x\|_{p;[0,T]} < \infty$.

Now the *Skorohod problem* associated with a continuous path $w \in C([0,T], \mathbb{R}^d)$ is to find (ξ, ϕ) such that for all $0 \leq t \leq T$

$$\xi_t = w_t + \phi(t), \quad \xi_t \in \overline{D},$$

$$\phi(t) = \int_0^t \mathbf{1}_{\partial D}(\xi_s)\mathbf{n}(s) d\|\phi\|_{1;[0,s]}, \quad \mathbf{n}(s) \in \mathcal{N}_{\xi_s} \text{ if } \xi_s \in \partial D. \tag{1}$$

10.1.1 A basic example

The most basic example is the case $d = 1, D = [0, \infty)$. This is the original problem considered by Skorohod. Here we interpret the value ξ_t as the position of a particle at time t. Then the Skorohod problem asks for the particle's trajectory under the following constraints. First, it must always remain within the given domain. It must never cross the boundary, that is in our case $\xi_t \geq 0$. Second, the particle should move according to the function w (except when it would otherwise cross the boundary). That is, the increments of ξ should equal the increments of w whenever $\xi_t > 0$. The precise way to express this constraint is the equation (1). Here the function ϕ should be interpreted as compensating for the times when w is trying to lead ξ out of the domain. In our basic example, the equation (1) reads as

$$\phi(t) = \int_0^t \mathbf{1}_{\xi_s=0}\, d\phi(s)$$

This is equivalent to

$$0 = \int_0^T \mathbf{1}_{(0,\infty)}(\xi_s) d\phi(s)$$

which means precisely that the compensation function is not supposed to change whenever $\xi_s > 0$. In this one-dimensional example, the solution to the Skorohod problem is very simple. It is given by the function

$$\xi_t = w_t - \min_{0 \leq s \leq t} \min\{w_s, 0\}.$$

In higher dimensions, it is a more delicate issue and the existence and uniqueness depend on properties of the boundary of the domain. For further reading, see the exposition [4].

10.2 Main results

We are going to consider the following conditions on the boundary of D.

(A) There exists $r_0 > 0$ such that we have

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$$

for all $x \in \partial D$.

(B) There exist $\delta > 0$ and $\beta \geq 1$ such that for every $x \in \partial D$ there exists a unit vector l_x satisfying

$$\langle l_x, \mathbf{n} \rangle \geq \frac{1}{\beta}$$

for all $\mathbf{n} \in \bigcup_{y \in B(x,\delta) \cap \partial D} \mathcal{N}_y$.

If the assumptions (A) and (B) are satisfied, then the Skorohod problem considered above has a unique solution.

A continuous map $\omega : \{(s,t) : 0 \leq s \leq t \leq T\} \to [0, \infty)$ satisfying

$$\omega(s,t) + \omega(t,u) \leq \omega(s,u)$$

for all $s \leq t \leq u$ is called a *control function* for $(x_t)_{t \in [0,T]}$ if there exists a constant $0 < C < \infty$ such that for every $0 \leq s \leq t \leq T$ we have

$$|x_t - x_s| \leq C\omega(s,t).$$

We can now formulate the first main result.

Theorem 1. *Let x_t for $0 \leq t \leq T$ be a continuous path on $\mathbb{R}^n$ that is of finite p-variation for some $1 \leq p < 2$. Let $\omega(s,t)$ be a control function of x_t and $\sigma \in C_b^1(\mathbb{R}^d, \mathbb{R}^n \otimes \mathbb{R}^d)$, $y_0 \in \overline{D}$. Assume that (A) and (B) hold. Then there exists a solution (y, Φ) to the reflected differential equation,*

$$y_t = y_0 + \int_0^t \sigma(y_s) dx_s + \Phi(t).$$

The integral here is in the sense of Young [5]. Moreover, the solution satisfies

$$|y_t - y_s| \leq C\omega(s,t)^{1/p},$$

$$\|\Phi\|_{[s,t]} \leq C\omega(s,t)^{1/p}.$$

Here C is a constant that depends on $\omega(0,T)$, σ and the domain D.

The rough idea of the proof is to follow an Euler approximation argument in the spirit of Davie [3].

The second main result is concerned with reflected differential equations driven by p-rough paths, where $2 \leq p < 3$. For definition and an accessible exposition of p-rough paths and other basic topics of rough paths theory, see

[2]. Before stating the result we need some preliminary definitions. Let $2 \leq p < 3, 0 \leq s \leq t \leq T$ and $X_{s,t} = (1, X^1_{s,t}, X^2_{s,t}) \in \Omega_p(\mathbb{R}^n), Y_{s,t} = (1, Y^1_{s,t}, Y^2_{s,t}) \in \Omega_p(\mathbb{R}^d)$ be p-rough paths and $\Phi(t)$ a continuous bounded variation path on $\mathbb{R}^d$. Also, let $\sigma \in C^2_b(\mathbb{R}^d, \mathbb{R}^n \otimes \mathbb{R}^d)$. Then the pair (Y, Φ) is called a *solution* to the rough differential equation on D driven by X with normal reflection and starting point $y_0 \in \overline{D}$,

$$dY_t = \sigma(Y_t)dX_t + d\Phi(t), \qquad Y_0 = y_0, \tag{2}$$

if the following conditions hold (always $0 \leq s \leq t \leq T$):

1. Let $Y_t = y_0 + Y^1_{0,t}$. Then $Y_t \in \overline{D}$ and there exists a Borel measurable map $\mathbf{n} : [0, T] \to \mathbb{R}^d$ such that $\mathbf{n}(s) \in \mathcal{N}_{Y_s}$ if $Y_s \in \partial D$ and

   $$\Phi(t) = \int_0^t \mathbf{1}_{\partial D}(Y_s)\mathbf{n}(s)d\|\Phi\|_{1;[0,s]}$$

   for all $0 \leq t \leq T$.

2. $Y_{s,t}$ is a solution to the following rough differential equation:

   $$dY_t = \widehat{\sigma}(Y_t)d\widehat{X}_t, \qquad Y_0 = y_0,$$

   where $\widehat{\sigma}(x)$ is a linear map $\mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

   $$\widehat{\sigma}(x)(\xi, \eta) = \sigma(x)\xi + \eta$$

   and the driving rough path $\widehat{X} \in \Omega_p(\mathbb{R}^n \times \mathbb{R}^d)$ is given by

   $$\widehat{X}^1_{s,t} = (X^1_{s,t}, \Phi(t) - \Phi(s)),$$

   $$\widehat{X}^2_{s,t} = \Big(X^2_{s,t}, \int_s^t X^1_{s,u} \otimes d\Phi(u), \int_s^t (\Phi(u) - \Phi(s)) \otimes dX^1_{s,u}, \int_s^t (\Phi(u) - \Phi(s)) \otimes d\Phi(u)\Big).$$

The main theorem is as follows.

Theorem 2. *Let $\sigma \in C^3_b(\mathbb{R}^d, \mathbb{R}^n \otimes \mathbb{R}^d)$. Also, assume condition (A) and that the Skorohod problem has a unique solution (ξ, ϕ) for every w and denote $L(w) = \phi$. Moreover, suppose there exists a positive constant C_D such that for all continuous paths w on $\mathbb{R}^d$ we have*

$$\|L(w)\|_{1;[s,t]} \leq C_D\|w\|_{\infty;[s,t]}$$

for $0 \leq s \leq t \leq T$*. Let* ω *be the control function of* $X_{s,t}$*, i.e., we have that*

$$|X^i_{s,t}| \leq \omega(s,t)^{i/p}$$

for $0 \leq s \leq t \leq T$ *and* $i = 1, 2$*. Then there exists a solution* (Y, Φ) *to the reflected rough differential equation* (2) *such that for all* $0 \leq s \leq t \leq T$*,*

$$|Y^i_{s,t}| \leq C(1 + \omega(0,T))^3 \omega(s,t)^{i/p}, \quad i = 1, 2,$$

$$\|\Phi\|_{1;[s,t]} \leq C(1 + \omega(0,T))^3 \omega(s,t)^{1/p},$$

where $0 < C < \infty$ *is a constant that depends only on* σ, C_D *and* p*.*

A further result, which we will discuss if time allows, is concerned with the relation between the solution to a reflected rough differential equation and a reflected stochastic differential equation when the driving process is a Brownian motion.

References

[1] S. Aida, *Reflected rough differential equations.* Stochastic Process. Appl. 125 (2015), no. 9, 3570–3595.

[2] F. Baudoin, *Rough paths theory.* Lecture notes, 2013. Available under http://www.math.purdue.edu/∼fbaudoin/Rough.pdf.

[3] A. M. Davie, *Differential equations driven by rough paths: an approach via discrete approximations.* Appl. Math. Res. Express. AMRX 2007, no. 2, Art. ID abm009, 40 pp.

[4] A. Pilipenko, *An introduction to stochastic differential equations with reflection.* Lectures in Pure and Applied Mathematics, Potsdam University Press, 2014.

[5] L. C. Young, *An inequality of Hölder type, connected with Stieltjes integration.* Acta Math., 67 (1936), 251 – 282.

Joris Roos, University of Bonn
email: **jroos@math.uni-bonn.de**

11 Rough solutions for the periodic Korteweg-de Vries equation

after M. Gubinelli [1]
A summary written by Jonas Jansen

Abstract

We show how to apply ideas from the theory of rough paths to the analysis of low-regularity solutions to non-linear dispersive equations. Our basic example will be the one dimensional Korteweg-de Vries equation (KdV) on a periodic domain and with initial condition in $\mathcal{F}L^{\alpha,p}$ spaces.

11.1 Introduction

We want to solve the classical Korteweg-de Vries equation (KdV)

$$\partial_t u(t,\xi) + \partial_\xi^3 u(t,\xi) - \frac{1}{2}\partial_\xi (u(t,\xi)^2) = 0, \quad u(0,\xi) = u_0(\xi) \tag{1}$$

for $(t,\xi) \in [0,T^*] \times \mathbb{T}$ where $\mathbb{T} = [-\pi,\pi]$ is the torus and u_0 is of low regularity.

We denote with $\mathcal{F}f = \hat{f} \colon \mathbb{Z} \to \mathbb{C}$ the Fourier coefficients of a real function $f \colon \mathbb{T} \to \mathbb{R} : f(\xi) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ik\xi}$. Now we define the space $\mathcal{F}L^{\alpha,p}$ in which our initial condition will live:

$$\mathcal{F}L^{\alpha,p} := \{f \in \mathcal{S}'(\mathbb{T}) : \hat{f}(0) = 0, |f|_{\mathcal{F}L^{\alpha,p}} = |(1+|\cdot|^2)^{\frac{1}{2}} \hat{f}(\cdot)|_{\ell^p} < \infty\}$$

where $\mathcal{S}'(\mathbb{T})$ is the space of Schwartz distributions on the torus. Due to conservation laws, the restriction to mean zero function is a natural setting to discuss KdV and we remark $\mathcal{F}L^{\alpha,p} = H^\alpha/\mathbb{R}$.
Furthermore, given normed vector spaces V, W we denote by $\mathcal{L}(V,W)$ the Banach space of bounded linear operators and introduce the short-hand $\mathcal{L}^n V := \mathcal{L}(V^n, V)$. We use the convention that for $X \in \mathcal{L}^n V$, we write $X(a)$ instead of $X(a,\dots,a)$ for $a \in V$.

11.2 The sewing map

We want to use similar techniques as for RDEs when solving a fixed-point equation. So we introduce $\mathcal{C}_n V := \{a \in C([0,T]^n, V) : a_{t_1,\dots,t_n} = 0 \text{ when } t_i = t_j \text{ for } i \neq j\}$ and operators $\delta\colon \mathcal{C}_1 V \to \mathcal{C}_2 V : \delta f_{ts} = f_t - f_s$, $\delta\colon \mathcal{C}_2 V \to \mathcal{C}_3 V : \delta a_{tus} = a_{ts} - a_{tu} - a_{us}$. It is $\delta^2 = 0$ and if $a \in \mathcal{C}_2 V$ satisfies $\delta a = 0$, then there is $f \in \mathcal{C}_1 V$ with $\delta f = a$. Define $\mathcal{Z}\mathcal{C}_3 V = \mathcal{C}_3 V \cap \operatorname{im}\delta$. We introduce Hölder-like norms on $\mathcal{C}_n V$:

$$\|f\|_\mu := \sup_{t,s\in[0,T]} \frac{|f_{ts}|}{|t-s|^\mu}, \quad \mathcal{C}_2^\mu V := \{f \in \mathcal{C}_2 V : \|f\|_\mu < \infty\}$$

and setting $\|h\|_{\gamma,\rho} := \sup_{t,u,s\in[0,T]} \frac{|h_{tus}|}{|u-s|^\gamma |t-u|^\rho}$

$$\|h\|_\mu :=?? \inf\left\{\sum_{i=1}^n \|h_i\|_{\mu-\rho_i,\rho_i} : \sum_{i=1}^n h_i = h, h_i \in \mathcal{C}_3 V, n \in \mathbb{N}, \rho_i \in (0,\mu)\right\}$$

we define the space $\mathcal{C}_3^\mu V := \{h \in \mathcal{C}_3 V : \|h\|_\mu < \infty\}$. Define $\mathcal{C}_n^{1+} V := \bigcup_{\gamma>1} \mathcal{C}_n^\gamma V$. Use the norms and notations also for $\mathcal{Z}\mathcal{C}_3 V$. Then we have the following well-known Theorem (cp. [1], [3],[5]):

Theorem 1. *There is a unique linear map* $\Lambda\colon \mathcal{Z}\mathcal{C}_3^{1+} V \to \mathcal{C}_2^{1+} V$ *such that* $\delta\Lambda = \mathrm{Id}_{\mathcal{Z}\mathcal{C}_3 V}$ *and for any* $\mu > 1$, $\Lambda\colon \mathcal{Z}\mathcal{C}_3^\mu V \to \mathcal{C}_2^\mu V$ *satisfies*

$$\|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2}\|h\|_\mu.$$

11.3 The equation

Using the Duhamel principle, we can rewrite equation (1) via the fixed-point problem

$$u(t) = U(t)u_0 + \int_0^t U(t-s)\mathcal{N}(u(s))ds \tag{2}$$

where $\mathcal{N}(u(t,\xi)) := \frac{1}{2}\partial_\xi(u(t,\xi)^2)$ and $U(t)$ is the Airy group associated with the linear problem $\partial_t u + \partial_\xi^3 u = 0, ?? \quad u(0,\cdot) = u_0 \in \mathcal{F}L^{\alpha,p}$. $U(t)$ is acting on $\mathcal{F}L^{\alpha,p}$ by isometries and it is $\mathcal{F}(U(t)\phi)(k) = e^{ik^3 t}\hat{\phi}(k)$. Changing variables $v(t) = U(-t)u(t)$ we find

$$v(t) = v_0 + \int_0^t \dot{X}_s(v(s))ds \tag{3}$$

where $v_0 = u_0$ and $\dot{X}_s \in \mathcal{L}^2 \mathcal{F}L^{\alpha,p}$ is given by $\dot{X}_s(\phi_1, \phi_2) = U(-s)\partial_\xi[(U(s)\phi_1)(U(s)\phi_2)]/2$.

Now, formally, we expand the integral part:

$$\begin{aligned}\int_0^t \dot{X}_s(v(s))ds &= \int_0^t \dot{X}_s(v_0)ds + \int_0^t \dot{X}_s(v(s)) - \dot{X}_s(v_0)ds = \int_0^t \dot{X}_s(v(s)) - \dot{X}_s(v_0)ds \\ &= \int_0^t \int_0^s \partial_\sigma \dot{X}_s(v(\sigma))d\sigma ds = 2\int_0^s \dot{X}_s(\partial_\sigma v(\sigma), v(\sigma))d\sigma ds.\end{aligned}$$

This leads to the power series expansion

$$v(t) = v_0 + \int_0^t \dot{X}_s(v_0)ds + 2\int_0^t \int_0^s \dot{X}_s(v_0, \dot{X}_\sigma(v_0))d\sigma ds + h.o.t. \tag{4}$$

We are not trying to show convergence of this series but rather use methods from rough paths writing

$$\delta v_{ts} = X_{ts}(v_s) + r_{ts} \tag{5}$$

where r_{ts} is a remainder that we hope to be in some sense small and X_{ts} is given by the first-order term of the power series expansion, i.e.

$$\mathcal{F}X_{ts}(\phi_1, \phi_2)(k) = \int_s^t \mathcal{F}\dot{X}_\sigma(\phi_1, \phi_2)(k)d\sigma.$$

You can immediately check that $\delta X_{tus} = 0$ and hence using bilinearity $\delta r_{tus} = X_{tu}(\delta v_{us}, v_s) + X_{tu}(v_u, \delta v_{us})$. Using the sewing map, we would like to solve the fixed-point equation

$$\delta v = X(v, v) + \Lambda[X(\delta v, v) + X(v, \delta v)]$$

provided $X(\delta v, v) + X(v, \delta v) \in \mathcal{Z}\mathcal{C}_3^{1+}V$. As we are not able to prove this, we go to second order, obtaining the equation

$$\delta v_{ts} = X_{ts}(v_s) + X^2_{ts}(v_s) + r^2_{ts}$$

where $X^2 \in \mathcal{C}_2\mathcal{L}^3\mathcal{F}L^{\alpha,p}$ is given by

$$X^2_{ts}(\phi_1, \phi_2, \phi_3) = 2\int_s^t d\sigma \int_s^{\sigma_1} \dot{X}_\sigma(\phi_1, \dot{X}_{\sigma_1}(\phi_2, \phi_3)).$$

Computing δr^2_{tus}, we obtain the second-order Λ-equation

$$\delta v = X(v) + X^2(v) + v^\flat \tag{6}$$

with $v^\flat_{ts} = \Lambda[2X_{tu}(\delta v_{us} - X_{us}(v_s), v_s) + X_{tu}(\delta_{us} v, \delta v_{us}) + X^2_{tu}(\delta v_{us}, v_s, v_s) + X^2_{tu}(v_u, \delta v_{us}, v_s) + X^2_{tu}(v_u, v_u, \delta v_{us})]_{ts}$.

For this equation to be well-defined, we need $2X(\delta v - X(v), v) + X(\delta v, \delta v) + X^2(\delta v, v, v) + X^2(v, \delta v, v) + X^2(v, v, \delta v)$ to be in the domain of Λ. Therefore, we first establish bounds on X, X^2 respectively: given $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}_+$ by $\mathcal{D}_p := \{(\gamma, \alpha) : \alpha \geq -\frac{1}{2} - \frac{1}{p} + \gamma, \gamma \in [0, \frac{1}{4})\} \cup \{(\gamma, \alpha) : \alpha > -1 - \frac{1}{p} + 3\gamma, \gamma \in [\frac{1}{4}, \frac{1}{2})\}$ if $p > 2$ and $\mathcal{D}_p := \{(\gamma, \alpha) : \alpha \geq -1 + \gamma, \gamma \in [0, \frac{1}{2p})\} \cup \{(\gamma, \alpha) : \alpha > -1 - \frac{1}{p} + 3\gamma, \gamma \in [\frac{1}{2p}, \frac{1}{2})\}$ if $1 \leq p \leq 2$, the following result gives a quantitative control on X_{ts}:

Lemma 2. *For any $(\gamma, \alpha) \in \mathcal{D}_p$ $X_{ts} \in \mathcal{L}^2 \mathcal{F}L^{\alpha,p}$ and $|X_{ts}|_{\mathcal{L}^2 \mathcal{F}L^{\alpha,p}} \lesssim |t-s|^\gamma$.*

We also have a bound for X^2 on $\mathcal{D}'_p := \mathcal{D}_p \cap \{\alpha > \alpha_*(p)\} := \max\{-\frac{1}{p}, -\frac{1}{2}\}$. Remark that X^2 is an unbounded operator.

Lemma 3. *There exist unbounded linear operators $\hat{X}^2_{ts}, \check{X}^2_{ts} \colon (\mathcal{F}L^{\alpha,p})^3 \to \mathcal{F}L^{\alpha,p}$, $X^2_{ts} = \hat{X}^2_{ts} + \check{X}^2_{ts}$, $\delta \check{X}^2 = 0$, such that when $\alpha > \alpha_*(p)$, it is $|\check{X}^2_{ts}|_{\mathcal{L}^3 \mathcal{F}L^{\alpha,p}} \lesssim |t-s|$ and for any $(\gamma, \alpha) \in \mathcal{D}$ it is $|\check{X}^2_{ts}|_{\mathcal{L}^3 \mathcal{F}L^{\alpha,p}} \lesssim |t-s|^{2\gamma}$.*

11.4 The main result

We fix the requirements for a solution:

$$\sup_{t \in [0,T]} |v_t|_{\mathcal{F}L^{\alpha,p}} < \infty, \quad \delta v \in \mathcal{C}^\gamma_1 \mathcal{F}L^{\alpha,p}, ?? \quad \delta v - X(v,v) \in \mathcal{C}^{2\gamma}_2 \mathcal{F}L^{\alpha,p}.$$

Then using Lemmas (2) and (3), it is easy to check that $v^\flat$ is well-defined via the sewing map if $3\gamma > 1$. So remark that for every $\alpha > \alpha_*(p)$ and every $p \in [1, \infty]$, there is a pair $(\gamma, \alpha) \in \mathcal{D}'_p$ with $\gamma > \frac{1}{3}$.

Solving equation (6) we use a standard fixed-point argument involving a contraction. So the last bit is to define a space on which to define the contraction mapping. For $0 < \eta \leq \gamma$, define

$$\mathcal{Q}_\eta := \{(y, y', y^\#) : y, y' \in \mathcal{C}^\eta_1 \mathcal{F}L^{\alpha,p}, y^\# \in \mathcal{C}^{2\eta}_2 \mathcal{F}L^{\alpha,p}\}$$

together with the metric

$$d_{\mathcal{Q},\eta}(y,z) := |y_0 - z_0| + \|y - z\|_\eta + \|y^\# - z^\#\|_\eta.$$

Now, for $\eta < \gamma$ we want to find a fixed-point of the map $\Gamma\colon \mathcal{Q}_\eta \to \mathcal{Q}_\eta$: $\Gamma(y, y', y^{\#}) = (z, z', z^{\#})$ with

$$z_0 = y_0, \quad \delta z = X(y, y) + X^2(y, y, y) + z^\flat$$

for $z^\flat = \Lambda[2X(y^{\#}, y) + X(\delta y, \delta y) + X^2(\delta y, y, y) + X^2(y, \delta y, y) + X^2(y, y, \delta y)]$. Further we set $z' = y$, $z^{\#} = X^2(y, y, y) + z^\flat$. If $\eta > \frac{1}{3}$, then $z^\flat$ is well-defined, as all arguments of the sewing map belong to $\mathcal{C}_3^{3\eta}\mathcal{F}L^{\alpha,p}$. Now for T sufficiently small, $\Gamma(y) \in \mathcal{Q}_\gamma$ and Γ is a contraction. Thus we obtain the following Theorem.

Theorem 4. *For any $\alpha > \alpha_*(p)$, $(\gamma, \alpha) \in \mathcal{D}'_p$ $\gamma > \frac{1}{3}$, any $v_0 \in \mathcal{F}L^{\alpha,p}$ there is $T^* = T^*(|v_0|_{\mathcal{F}L^{\alpha,p}}) > 0$ sufficiently small and a unique $v \in \mathcal{C}^\gamma\mathcal{F}L^{\alpha,p}$ such that $v(0) = v_0$ and*

$$v_t = v_s + X_{ts}(v_s) + X^2_{ts}(v_s) + o(|t - s|) \quad \forall 0 \le s \le t \le T^*.$$

The solution map $\Theta\colon \mathcal{C}_2^\gamma\mathcal{L}^2\mathcal{F}L^{\alpha,p} \times \mathcal{C}_2^{2\gamma}\mathcal{L}^3\mathcal{F}L^{\alpha,p} \times \mathcal{F}L^{\alpha,p} \to \mathcal{Q}_\gamma : v = \Theta(X, X^2, v_0)$ is locally Lipschitz.

As in the finite-dimensional theory of RDEs, given partitions $\Pi = \{0 = t_0 < \ldots < t_n = t\}$, we can use the approximation by Riemann sums as in [2] to write

$$v_t = v_0 + \lim_{|\Pi| \to 0} \sum_i X_{t_{i+1}t_i}(v_{t_i}) + X^2_{t_{i+1}t_i}(v_{t_i}).$$

It remains to connect the solution of the Λ-equation (6) with solutions of KdV. Therefore denote by $(P_N f)(\xi) = \sum_{|k| \le N} \hat{f}(k)e^{ik\xi}$ the Fourier projector. Defining $u(t) = U(t)v(t)$ where v is a solution as in Theorem (4), $P_N u$ is smooth and we have the following Corollary.

Corollary 5. *$\mathcal{N}(P_N u)$ converges in $C([0, T], \mathcal{S}'(\mathbb{T}))$ to a limit we denote by $\mathcal{N}(u)$, i.e. for any $\phi \in \mathcal{S}(\mathbb{T})$ it is $\sup_{t\in[0,T]} |\mathcal{N}(P_N u)(t, \phi) - \mathcal{N}(u)(t, \phi)| \to 0$ as $N \to \infty$. Furthermore $\mathcal{N}(u)$ satisfies the distributional equation*

$$\partial_t z + \partial_\xi^3 z - \mathcal{N}(u) = 0.$$

11.5 Applications

Due to keep this summary short I will not go into detail concerning various applications of the methods introduced. By an L^2-conservation law, you can show that if the initial condition v_0 is in L^2 then the solution is global.

Furthermore there exist applications to show convergence of numerical schemes. As shown in [6], the solutions of the approximate KdV

$$\partial_t u^{(N)} + \partial_\xi^3 u^{(N)} - P_N \mathcal{N}(u^{(N)}) = 0, \quad u^{(N)}(0) = P_N u_0$$

do not even weakly converge to a solution of KdV. But inspired by rough path analysis and using the methods introduced above you can show that a modified scheme

$$\partial_t u^{(N)} + \partial_\xi^3 u^{(N)} - P_N \mathcal{N}(u^{(N)}) - \Gamma^{(N)}(u^{(N)}) = 0, \quad u^{(N)}(0) = P_N u_0$$

is converging to a solution of KdV.

Another application is given by a similar result as in Theorem (4) of unique pathwise solutions in $C^\gamma \mathcal{F}L^{\alpha,p}$ for the KdV-equation with presence of an additive random force

$$\partial_t u + \partial_\xi^3 u - \frac{1}{2}\partial_\xi(u^2)??\Phi\partial_t\partial_\xi B, \quad u(0) = u_0 \in \mathcal{F}L^{\alpha,p}$$

where $\partial_t\partial_\xi B$ is white noise on $\mathbb{R} \times \mathbb{T}$ and Φ is a linear operator acting diagonally on ξ in Fourier space.

References

[1] Gubinelli, M., *Rough solutions for the periodic Korteweg-de Vries equation.* arXiv:math/0610006 [math.AP] (2009);

[2] Gubinelli, M., *Controlling rough paths.* Funct. Anal., 216(1) (2004), pp. 86-140.

[3] Gubinelli, M. and Tindel, S., *Rough evolution equations.* The Annals of Probability, Vol. 38, No. 1 (2010), pp. 1-75.

[4] Lyons, T. J., *Differential equations driven by rough signals.* Rev. Mat. Iberoamericana, 14(2) (1998), pp. 215-310.

[5] Friz, P. and Hairer, M., *A Course on Rough Paths.* Springer (2014).

[6] Colliander, J., Keel, M., Staffilani, G., Takaoka, H and Tao, T. *Symplectic nonsqueezing of the Korteweg-de Vries flow.* Acta Math. 195 (2005), pp. 197-252.

Jonas Jansen, University of Bonn
email: **s6jojans@uni-bonn.de**

12 Deterministic homogenization for fast-slow systems with chaotic noise

after D. Kelly and I. Melbourne [1]
A summary written by Johanna Richter

Abstract

We consider a smooth, deterministic flow $(\phi_t)_{t\geq 0}$ on a finite dimensional manifold and give conditions under which the solutions x_ε of the ordinary differential equation $\dot{x}_\varepsilon = a(x_\varepsilon, y_\varepsilon) + \varepsilon^{-1} b(x_\varepsilon, y_\varepsilon), y_\varepsilon(t) = \phi_{\varepsilon^{-2}t} y_0$ converge weakly to an Itô diffusion X as $\varepsilon \to 0$.

12.1 Introduction

Let $(\phi_t)_{t\geq 0}$ be a smooth flow on a finite dimensional manifold M. Suppose that $\Omega \subset M$ is a closed flow-invariant set with ergodic probability measure μ. Define the random variable $y(t) = y(t, y_0) = \phi_t y_0$ on the probability space (Ω, μ), and set $y_\varepsilon(t) = y(\varepsilon^{-2} t)$ for $\varepsilon > 0$. Further, let $a, b : \mathbb{R}^d \times M \to \mathbb{R}^d$ be two vector fields and let the random variable x_ε on Ω be defined as the solution to the ODE

$$\frac{dx_\varepsilon}{dt} = a(x_\varepsilon, y_\varepsilon) + \frac{1}{\varepsilon} b(x_\varepsilon, y_\varepsilon), \quad x_\varepsilon(0) = \xi \tag{1}$$

with deterministic initial condition $\xi \in \mathbb{R}^d$. If a and b are suitably regular, x_ε takes values in the space of continuous functions $C([0, T], \mathbb{R}^d)$ for some finite $T > 0$. The aim is to identify the limiting behaviour of x_ε on $C([0, T], \mathbb{R}^d)$ as $\varepsilon \to 0$.

12.2 Main result

Definition 1. *For $u : \mathbb{R}^d \times M \to \mathbb{R}$ and $\alpha \in [0, \infty), \kappa \in [0, 1)$ define the mixed Hölder norm*

$$\|u\|_{C^{\alpha,\kappa}} = \sum_{|k|\leq \lfloor \alpha \rfloor} \sup_{x\in\mathbb{R}^d} \|\partial_x^k u(x, \cdot)\|_{C^\kappa} + \sum_{|k|=\lfloor \alpha \rfloor} \sup_{x,z\in\mathbb{R}^d} \frac{\|\partial_x^k u(x, \cdot) - \partial_x^k u(z, \cdot)\|_{C^\kappa}}{|x - z|^{\alpha - \lfloor \alpha \rfloor}},$$

where the second summation is omitted when α is an integer and $\|\cdot\|_{C^\kappa}$ is the standard Hölder norm acting on the y component. If u is vector-valued, we define $\|u\|_{C^{\alpha,\kappa}} = \sum_i \|u^i\|_{C^{\alpha,\kappa}}$. We write $u \in C^{\alpha,\kappa}(\mathbb{R}^d \times M, \mathbb{R}^d)$ if $\|u\|_{C^{\alpha,\kappa}} < \infty$.

In the following we say $u \in C_0^{\alpha,\kappa}$ if $u \in C^{\alpha,\kappa}$ and $\int_\Omega u(x,y)\, d\mu(y) = 0$, and we write $u \in C^{1+,\kappa}$ if there exists $\alpha > 1$ such that $u \in C^{\alpha,\kappa}$. Furthermore, we assume that the flow ϕ_t has the following properties.
Assumption A. For $v \in C_0^\kappa(\Omega, \mathbb{R}^m), \kappa > 0$ define

$$W_{v,n}(t) = n^{-1/2} \int_0^{tn} v \circ \phi_s \, ds \quad \text{and} \quad \mathbb{W}_{v,n}(t) = n^{-1} \int_0^{tn} \int_0^s v \circ \phi_r \otimes v \circ \phi_s \, dr ds.$$

Assume there is a bilinear operator $\mathfrak{B} : C_0^\kappa(\Omega, \mathbb{R}) \times C_0^\kappa(\Omega, \mathbb{R}) \to \mathbb{R}$ such that for all $v \in C_0^\kappa(\Omega, \mathbb{R}^m)$,

$$(W_{v,n}, \mathbb{W}_{v,n}) \to (W_v, \mathbb{W}_v)$$

as $n \to \infty$, in the sense of finite dimensional distributions, where W_v is a Brownian motion in $\mathbb{R}^m$ and $\mathbb{W}_v$ is the process with values in $\mathbb{R}^{m \times m}$ defined by

$$\mathbb{W}_v^{ij}(t) = \int_0^t W_v^i dW_v^j + \mathfrak{B}(v^i, v^j)t.$$

Assumption B. For $v, w \in C_0^\kappa(\Omega, \mathbb{R}), \kappa > 0$ define

$$v_t = \int_0^t v \circ \phi_s \, ds \quad \text{and} \quad S_t = \int_0^t \int_0^s v \circ \phi_r w \circ \phi_s \, dr ds.$$

Assume there is a $p > 3$, and for all $v, w \in C_0^\kappa(\Omega, \mathbb{R})$ there is $K = K(v, w, p) > 0$ such that for all $t \geq 0$

$$(\mathbb{E}_\mu |v_t|^{2p})^{1/2p} \leq K t^{1/2} \quad \text{and} \quad (\mathbb{E}_\mu |S_t|^p)^{1/p} \leq K t.$$

Now we state the main result.

Theorem 2. *Suppose that Assumptions A and B hold with some $p \in (3, \infty]$ and $\kappa > 0$. Further let $a \in C^{1+,0}(\mathbb{R}^d \times M, \mathbb{R}^d)$ and $b \in C_0^{\alpha,\kappa}(\mathbb{R}^d \times M, \mathbb{R}^d)$ for some $\alpha > 2 + \frac{2}{p-1} + \frac{d}{p}$. Then*

(i) The limit

$$\mathfrak{B}(v, w) = \lim_{n \to \infty} n^{-1} \int_\Omega \int_0^n \int_0^s v \circ \phi_r w \circ \phi_s \, dr ds d\mu$$

exists for all $v, w \in C_0^\kappa(\Omega, \mathbb{R})$ and the resulting bilinear operator $\mathfrak{B} : C_0^\kappa(\Omega, \mathbb{R}) \times C_0^\kappa(\Omega, \mathbb{R}) \to \mathbb{R}$ is bounded and positive semidefinite.

(ii) The drift and diffusion coefficients given by

$$\tilde{a}^i(x) = \int_\Omega a^i(x,y)d\mu(y) + \sum_{k=1}^{d} \mathfrak{B}(b^k(x,\cdot), \partial_{x_k} b^i(x,\cdot)), \quad i = 1,\dots,d,$$

$$(\sigma(x)\sigma(x)^T)^{ij} = \mathfrak{B}(b^i(x,\cdot), b^j(x,\cdot)) + \mathfrak{B}(b^j(x,\cdot), b^i(x,\cdot)), \quad i,j = 1,\dots,d,$$

are Lipschitz.

(iii) The family of solutions x_ε to the ODE (1) converges weakly in the supnorm topology to the unique solution X of the SDE

$$dX = \tilde{a}(X)dt + \sigma(X)dB, \quad X(0) = \xi, \tag{2}$$

where B is a standard Brownian motion in $\mathbb{R}^d$.

(iv) Let $v, w \in C_0^\kappa(\Omega, \mathbb{R})$. If the integral $\int_0^\infty \int_\Omega v\, w \circ \phi_t \, d\mu dt$ exists, then

$$\mathfrak{B}(v,w) = \int_0^\infty \int_\Omega v\, w \circ \phi_t \, d\mu dt.$$

The strategy is to prove a localized version of Theorem (2). Then, the full result follows via a smooth cutoff of the drift and diffusion coefficients of the SDE (2) and the repeated application of the portmanteau lemma.

12.3 Localized version

We say that $a : \mathbb{R}^d \times M \to \mathbb{R}^d$ has compact support, if there exists an $E > 0$ such that $a(x,y) = 0$ for any $|x| > E$ and $y \in M$.

Theorem 3. *Suppose that Assumptions A and B hold with some $p \in (3,\infty]$ and $\kappa > 0$. Further, let $a \in C^{1+,0}(\mathbb{R}^d \times M, \mathbb{R}^d), b \in C_0^{\alpha,\kappa}(\mathbb{R}^d \times M, \mathbb{R}^d)$ for some $\alpha > 2 + \frac{2}{p-1} + \frac{d}{p}$, and suppose that a, b have compact support. Then the conclusions from Theorem 2 hold.*

The proof of Theorem 3 is split in three steps and makes use of rough path theory, especially the theory of rough differential equations (RDEs). One essential ingredient of the proof is the following result on existence, uniqueness and continuous dependence of solutions to RDEs, see [2], Section 8.

Theorem 4. *Let $\mathcal{A}, \mathcal{B}$ be Banach spaces. Suppose that $V_\varepsilon : [0,T] \to \mathcal{A}, W_\varepsilon : [0,T] \to \mathcal{B}$ are smooth paths, and that $\mathbb{W}_\varepsilon(t) = \int_0^t W_\varepsilon \otimes dW_\varepsilon$ is the iterated Itô integral of W_ε. Let $\gamma \in (\frac{1}{3}, \frac{1}{2}]$ and $F \in C^{1+}(\mathbb{R}^d, \mathcal{L}(\mathcal{A}, \mathbb{R}^d)), H \in C^{\frac{1}{\gamma}+}(\mathbb{R}^d, \mathcal{L}(\mathcal{B}, \mathbb{R}^d))$, where $\mathcal{L}$ denotes the space of continuous linear functionals. Moreover, suppose that X_ε solves the ODE*

$$dX_\varepsilon = F(X_\varepsilon)dV_\varepsilon + H(X_\varepsilon)dW_\varepsilon, \quad X_\varepsilon(0) = \xi. \tag{3}$$

If $(V_\varepsilon, W_\varepsilon, \mathbb{W}_\varepsilon) \to_w (V, W, \mathbb{W})$ in the space $\mathscr{C}^{\beta,\gamma}$ of (β, γ)-rough paths for all $\beta \in (\frac{1}{2}, 1)$, then $X_\varepsilon \to_w X$ in the supnorm topology, where X solves the RDE

$$dX = F(X)dV + H(X)d\mathbf{W}, \quad X(0) = \xi, \tag{4}$$

with $\mathbf{W} = (W, \mathbb{W})$.

The steps of the proof of Theorem 3 are

1. Reformulation of x_ε into a rough path framework and showing that x_ε solves an ODE of the form (3).

2. Showing that x_ε converges weakly to the solution X of a RDE of the form (4).

3. Rewriting the RDE (4) as the desired Itô SDE (2).

12.3.1 Rough path reformulation of the ODE (1)

Take $\mathcal{A} = C^{1+}(\mathbb{R}^d, \mathbb{R}^d), \mathcal{B} = C^\theta(\mathbb{R}^d, \mathbb{R}^d)$ with $\theta > 2 + \frac{2}{p-1}$ and define the smooth paths

$$V_\varepsilon(t) = \int_0^t a(\cdot, y_\varepsilon(r))dr \quad \text{and} \quad W_\varepsilon(t) = \varepsilon^{-1} \int_0^t b(\cdot, y_\varepsilon(r))dr, \quad t \in [0,T],$$

and the multidimensional Dirac distribution operators $F : \mathbb{R}^d \to \mathcal{L}(\mathcal{A}, \mathbb{R}^d)$ and $H : \mathbb{R}^d \to \mathcal{L}(\mathcal{B}, \mathbb{R}^d)$ by setting $H(x)(u) = u(x)$. One proves by plugging in the definitions

Lemma 5. *Suppose that a and b are as in Theorem 3 and let $V_\varepsilon, W_\varepsilon, F, H$ be defined as above. Then $F \in C^{1+}(\mathbb{R}^d, \mathcal{L}(\mathcal{A}, \mathbb{R}^d)), H \in C^\theta(\mathbb{R}^d, \mathcal{L}(\mathcal{B}, \mathbb{R}^d))$ and the solution x_ε of the ODE (1) satisfies the ODE*

$$dx_\varepsilon = F(x_\varepsilon)dV_\varepsilon + H(x_\varepsilon)dW_\varepsilon, \quad x_\varepsilon(0) = \xi. \tag{5}$$

12.3.2 Convergence to the RDE (4)

Let $\mathbb{W}_\varepsilon : [0,T] \to \mathcal{B} \otimes \mathcal{B}$ be the iterated integral of W_ε

$$\mathbb{W}_\varepsilon(t) = \int_0^t W_\varepsilon \otimes dW_\varepsilon = \varepsilon^{-2} \int_0^t \int_0^r b(\cdot, y_\varepsilon(u)) \otimes b(\cdot, y_\varepsilon(r)) du dr,$$

set $\mathbf{W}_\varepsilon = (W_\varepsilon, \mathbb{W}_\varepsilon) : [0,T] \to \mathcal{B} \times (\mathcal{B} \otimes \mathcal{B})$, and define the deterministic element $V \in C^1([0,T], \mathcal{A})$ by $V(t) = \bar{a}t$ with

$$\bar{a} = \int_\Omega a(\cdot, y) d\mu(y) \in \mathcal{A}.$$

Theorem 6. *Assume the set up of Theorem 3. Then for any $\beta \in (\frac{1}{2}, 1)$ and $\gamma \in (\frac{1}{3}, \frac{1}{2} - \frac{1}{2p})$*

(a) the family $(V_\varepsilon, \mathbf{W}_\varepsilon)_{\varepsilon>0}$ is tight in $\mathscr{C}^{\beta,\gamma}$.

(b) $V_\varepsilon \to_w V$ in the space $\mathscr{C}^\beta$ of β-rough paths.

(c) the family $(x_\varepsilon)_{\varepsilon>0}$ is tight in $C([0,T], \mathbb{R}^d)$, and every limit point X satisfies the RDE

$$dX = F(X)\bar{a}dt + H(X)d\mathbf{W}, \quad X(0) = \xi, \tag{6}$$

where $\mathbf{W}$ is a limit point of $(\mathbf{W}_\varepsilon)_{\varepsilon>0}$ in the space $\mathscr{C}^\gamma$ of γ-rough paths.

Idea of the Proof. (a) Apply a standard Arcela-Ascoli argument in combination with estimates on the $\mathscr{C}^{\beta,\gamma}$-norm, that are based on the embedding of Hölder spaces into Besov spaces.
(b) By ergodicity of μ it follows that $\pi V_\varepsilon \to \pi V$ in probability in $C([0,T], \mathbb{R})$ for every $\pi \in \mathcal{L}(\mathcal{A}, \mathbb{R})$. Proof by contradiction gives convergence in probability in $\mathscr{C}^{\beta,\gamma}$, which in turn implies weak convergence.
(c) Assume (a) and apply Prokhorov's theorem. After passing to a subsequence, we get weak convergence of $(V_\varepsilon, \mathbf{W}_\varepsilon)$ in $\mathscr{C}^{\beta,\gamma}$. Application of Theorem 4 and Lemma 5 in combination with (b) proves the claim.

12.3.3 Rewriting the RDE (4) as the SDE (2)

Let us partly characterize the limit points of $(\mathbf{W}_\varepsilon)_{\varepsilon>0}$ first. Take an arbitrary $\pi \in \mathcal{L}(\mathcal{B}, \mathbb{R}^m)$. Then it is $\pi b \in C_0^\kappa(\Omega, \mathbb{R}^m)$ and after a change of variables we have

$$(\pi W_\varepsilon, (\pi\otimes\pi)\mathbb{W}_\varepsilon) = \left(\varepsilon \int_0^{\varepsilon^{-2}t} (\pi b)\circ\phi_s ds, \varepsilon^2 \int_0^{\varepsilon^{-2}t}\int_0^s (\pi b)\circ\phi_r \otimes (\pi b)\circ\phi_s dr ds\right).$$

By Assumption A, we see that $(\pi W_\varepsilon, (\pi\otimes\pi)\mathbb{W}_\varepsilon) \to (W_{\pi b}, \mathbb{W}_{\pi b})$, as $\varepsilon\to 0$, in the sense of finite dimensional distributions, where $W_{\pi b}$ is a Brownian motion in $\mathbb{R}^m$ and $\mathbb{W}_{\pi b}$ is given by

$$\mathbb{W}^{ij}_{\pi b}(t) = \int_0^t W^i_{\pi b} dW^j_{\pi b} + \mathfrak{B}(\pi^i b, \pi^j b)t. \tag{7}$$

Lemma 7. *Let* $\mathbf{W}$ *be any limit point of* $(\mathbf{W}_\varepsilon)_{\varepsilon>0}$ *and let* X *be the solution to the RDE (4) driven by* $\mathbf{W}$*. Then* X *is a weak solution to the SDE (2).*

Idea of the Proof. By [3] it is sufficient to solve the martingale problem associated with the generator of the SDE (2). This is equivalent to showing that for any $\varphi\in C^2(\mathbb{R}^d,\mathbb{R})$ the process M^n_Δ given by

$$\begin{aligned} M^{n+1}_\Delta &= M^n_\Delta + \pi_1(X(t_n))W(t_n,t_{n+1}) + (\pi_2(X(t_n))\otimes\pi_3(X(t_n)))\mathbb{W}(t_n,t_{n+1}) \\ &\quad - \sum_{k=1}^d \mathfrak{B}(b^k(X(t_n),\cdot), \partial_k(b^i(X(t_n),\cdot)\partial_i\varphi(X(t_n)))(t_{n+1}-t_n), \end{aligned}$$

is a martingale with respect to the filtration $(\mathcal{F}_{t_n})_n$ generated by $\mathbf{W}:[0,T]\to\mathcal{B}\times(\mathcal{B}\otimes\mathcal{B})$. Here $\Delta=(t_n)_n$ is a partition of $[s,t]\subset[0,T]$, and $\pi(x) = (\pi_1(x),\pi_2(x),\pi_3(x))\in\mathcal{L}(\mathcal{B},\mathbb{R}^3)$ with $\pi_1(x)=\partial_i\varphi(x)H^i(x)$, $\pi_2(x)=H^k(x)$, $\pi_3=\partial_k(\partial_i\varphi(x)H^i(x))$. Thus, the martingale property of M^n_Δ is a consequence of the fact that

$$(\pi_1(x)W, (\pi_2(x)\otimes\pi_3(x))\mathbb{W}) \overset{dist}{=} (W_{\pi_1(x)b}, \mathbb{W}_{\pi_2(x)b\pi_3(x)b})$$

with $W_{\pi_1(x)b}$ a Brownian motion in $\mathbb{R}$ and $\mathbb{W}_{\pi_2(x)b\pi_3(x)b}$ defined analogous to (7).

Now we are ready for the
Proof of Theorem 3. By Theorem 6, any solution x_ε of the ODE (1) converges weakly along subsequences to a solution X of the RDE (4), which is a unique weak solution of the SDE (2) by Lemma 7. Thus, all subsequences converge to the same limit. The formula for $\mathfrak{B}$ follows by taking the expectation value in Assumption A and applying Assumption B to obtain convergence. □

References

[1] Kelly, D. and Melbourne, I., *Deterministic homogenization for fast-slow systems with chaotic noise.* Preprint (2014), arXiv:1409.5748v1.

[2] Friz, P. and Hairer, M., *A course on rough path.* Springer (to appear) (2014).

[3] Stroock, D. W., Varadhan, S. R. S., *Multidimensional diffussion processes.* Classics in Mathematics. Springer-Verlag, Berlin, 2006. Reprint of the 1997 edition.

Johanna Richter, University of Stuttgart
email: **johanna.richter@mathematik.uni-stuttgart.de**

13 A signed measure on rough paths associated to a PDE of high order: results and conjectures

after D. Levin, T Lyons [1].
A summary written by Gennady Uraltsev.

Abstract

This paper presents an attempt of defining a measure on the path space associated to the diffusion process governed by a higher order constant coefficient differential operator.

This approach is a generalization of the construction of the measure associated to the Laplacian, and thus to the heat semigroup, that yields the classical Wiener measure on the path space of the Brownian motion.

Generalizing this approach to higher order differential operators presents significant difficulties because for general differential operators the measure induced on the path space is not positive tends to have locally infinite total variation. Herein we construct a measure defined on piecewise linear paths subordinate to a given time partition.

We then show that when the mesh of the partitions go to zero the measures converge in a weak sense when evaluated against test functions consisting of iterated integrals of the paths.

In particular this allows one to calculate the expectation of the signature of a "random" path and suggests that the distributional limit may be actually be a measure on the path space of unparameterized paths (the path space quotiented by reparameterization.)

13.1 Motivation in the case of the Laplacian

Consider the heat equation on $[0,T]\times\mathbb{R}^d$ i.e. the diffusion equation associated with the Laplacian operator on $\mathbb{R}^d$:

$$\partial_t f(t,x) - \frac{1}{2}\Delta_x f(t,x) = 0.$$

It is a well known fact that the solution for initial data $f(0,x) = f_0(x) \in S(\mathbb{R})$ is given by the diffusion semigroup

$$f(t,x) = e^{\frac{t}{2}\Delta} f_0(x)$$

where the action of the semigroup is given by the convolution with a Schwartz kernel $\phi_t(x) := \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x|^2}{2t}}$:

$$S(t)f_0(x) = e^{\frac{t}{2}\Delta} f_0(x) = \int_{\mathbb{R}^d} f(x')\phi_t(x-x')dx'$$
$$S(t)S(s)f_0(x) = \phi_t * \phi_s * f_0(x) = \phi_{t+s} * f_0(x) = S(t+s)f_0(x) \qquad (1)$$

where the second entry encodes the semigroup property.

The kernel ϕ_t is a normalized positive function and as such is a density of a probability measure on $\mathbb{R}^d$. Kolmogorov's Extention Theorem allows us to define a probability measure on the product measure space $\left(\left(\mathbb{R}^d\right)^{[0,T]}, \mathcal{E}^{[0,T]}\right)$ that is naturally associated to the semigroup generated by setting

$$\mu(C_{t_1\ldots t_n}^{A_1\ldots A_n}) = \iiint_{A_1\times\cdots\times A_n} \phi_{t_1}(x_1)\phi_{t_2-t_1}(x_2-x_1)\ldots\phi_{t_n-t_{n-1}}(x_n-x_{n-1})dx_1\ldots dx_n \qquad (2)$$

for cylindrical sets

$$C_{t_1\ldots t_n}^{A_1\ldots A_n} := \left\{(x_t)_{t\in[0,T]} \in \left(\mathbb{R}^d\right)^{[0,T]} \,:\, x_{t_i} \in A_i \,\forall i \in \{1,\ldots,n\}\right\}.$$

Furthermore we can restrict the measure to the Wiener path space of continuous functions

$$C_0 = \{f \in C([0,T]) \,:\, f(0) = 0\}$$

thanks to Kolmogorov's Continuity Theorem that relies on some integrability conditions on the kernels ϕ_t. This construction effectively yields the Wiener measure, that is the law of the Brownian motion.

13.2 Generalities of higher order differential operators

A constant coefficient differential operator on $\mathbb{R}^d$ is an operator of the form

$$L := \sum_I a_I \partial^I = \sum_{I=(i_1\ldots,i_d)} a_{(i_1,\ldots,i_d)} \partial_1^{i_1}\ldots\partial_d^{i_d} \qquad a_I \in \mathbb{R}; \qquad (3)$$

we consider operators with real coefficients such that $a_{0\ldots0} = 0$ i.e. without the constant term. The degree of L is given by

$$\deg L := \max\left(|I| = |i_1| + \ldots |i_d| \,:\, a_I \neq 0\right).$$

Let $S(\mathbb{R}^d)$ be the set of Schwartz function and $f \mapsto \hat{f}$, $f \mapsto \check{f} : S(\mathbb{R}^d) \mapsto S(\mathbb{R}^d)$ be the Fourier transform and its inverse respectively. The operator L acts on $S(\mathbb{R}^d)$ as a multiplier operator

$$Lf(x) = \left(P(\xi)\hat{f}(\xi)\right)^{\vee}(x) \quad P(\xi) = \sum_I a_I (2\pi i\xi)^I = \sum_I a_I (2\pi i\xi_1)^{i_1} \dots (2\pi i\xi_d)^{i_d}.$$

We require weak growth condition on the multipliers.

Definition 1 (Minimal growth condition). *A polynomial $P(\xi)$ satisfies the minimal growth condition if*

$$P(\xi) \to +\infty \qquad \text{as } |\xi| \to \infty \tag{4}$$

Similarly to the case with the heat equation the solution of the diffusion equation

$$\begin{cases} \partial f(t,x) + Lf(t,x) = 0 & \text{on } [0,T] \times \mathbb{R}^d \\ f(0,x) = f_0(x) \in S(\mathbb{R}^d) \end{cases} \tag{5}$$

is given by the semigroup $t \in [0,T] \mapsto e^{-tL}$ operating continuously on $S(\mathbb{R}^d)$ via two equivalent definitions

$$e^{-tL}f(x) = \left(e^{-tP(\xi)}\hat{f}(\xi)\right)^{\vee}(x) \qquad e^{-tL}f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-t)^n L^n}{n!}. \tag{6}$$

The minimal growth condition (4) guarantees that for $t > 0$ the Schwartz kernel

$$\phi_t(x) = \left(e^{-tP(\xi)}\right)^{\vee} \in S(\mathbb{R}^d) \tag{7}$$

is actually an element of $S(\mathbb{R}^d)$ and thus the action of the semigroup is given by the convolution i.e. (1) still holds.

The crucial analytic result states that the semigroup e^{-tL} actually acts on all polynomials.

Proposition 2. *Suppose that L is a constant coefficient differential operator satisfying the minimal growth condition (4). Then the semigroup $t \mapsto e^{-tL}$ acts on the space of polynomials of degree up to $N \geq \deg L$ for an arbitrarily* *large N. In particular for all $t \in (0,T)$ and any polynomial $p(x)$ the pointwise identity*

$$\sum_{n=0}^{\infty} \frac{(-tL)^n}{n!} p(x) = \phi_t * p(x) := \int_{\mathbb{R}^d} p(x-x')\phi_t(x')dx'$$

holds and the generator of the action of the semigroup is $-L$ i.e.

$$\lim_{t \to 0} \frac{e^{-tL}p(x) - p(x)}{t} = -Lp(x)$$

where the limit is pointwise and in norm for any norm that is finite on polynomials up to a fixed degree.

13.3 Random paths and the main result

Unlike in the case of $L = -\frac{1}{2}\Delta$ the defining equality for the finite-dimensional marginal distributions cannot be extended to a measure on $\left(\mathbb{R}^d\right)^{[0,T]}$. Thus the approach to defining a measure on the path space via Kolmogorov's Extention Theorem is precluded.

We proceed by a cubature technique i.e. we approximate the measure by a sequence of measures supported on a simpler subset of the path space.

Let $D = (t_0, t_1, \ldots, t_n, t_{n+1})$ with $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ be a partition of the interval $[0,T]$ and we indicate by

$$\#D := \max_{i \in \{0,\ldots,n\}} |t_{i+1} - t_i|$$

the mesh of the partition. We work with the set of piecewise linear paths $PL_D[0,T]$ i.e. paths that are of the form

$$\gamma(t) = \frac{(t-t_i)\gamma_{i+1} - (t-t_{i+1})\gamma_i}{t_{i+1} - t_i} \qquad \text{for } t \in [t_i, t_{i+1}] \tag{8}$$

where $\gamma_i \in \mathbb{R}^d$ $i \in \{0,\ldots,n+1\}$. We denote the bijection from $\left(\mathbb{R}^d\right)^{n+1}$ to $PL_D[0,T]$ via $\pi_D : (\gamma_0, \ldots, \gamma_{n+1}) \mapsto \gamma$ with γ defined in (8). Equation (2) then defines a signed finite measure μ_D on $\left(\mathbb{R}^d\right)^n$ and we indicate

$$\mathbb{P}_D = \mu_D \circ \pi_D^{-1}$$

the signed finite measure on $PL_D[0,T]$.

The following weak convergence result result holds.

Theorem 3 (Expectation of the signature)**.** *Let $\mathbb{X}(\gamma)$ be the signature of a path γ then as we refine the partition D the following convergence result holds*

$$\lim_{\#D\to 0}\mathbb{E}(\mathbb{X}) = \lim_{\#D\to 0}\int_{PL_D[0,T]}\mathbb{X}(\gamma)d\mu_D(\gamma) = \exp\left(-\sum_{n=0}^{\infty} L(\underbrace{x\otimes\cdots\otimes x}_{n\ times})\right).$$

We recall that the signature of a path γ is an element $\mathbb{X}_{[0,T]}(\gamma)$ of the infinite tensor algebra

$$T(\mathbb{R}^d) = \bigoplus_{n=0}^{\infty}\left(\mathbb{R}^d\right)^{\otimes n}$$

given by the iterated integrals

$$\mathbb{X}_{[0,T]}(\gamma) = 1 + \sum_{n=1}^{\infty}\mathbb{X}^{(n)}_{[0,T]}(\gamma) \quad \mathbb{X}^{(n)}_{[0,T]}(\gamma) = \iiint_{\Delta_n[0,T]} d\gamma(t_1)\otimes\dots d\gamma(t_n) \tag{9}$$
$$\Delta_n[0,T] = \{(t_1,\dots,t_n)\in[0,T]^n : \, t_1<\cdots<t_n\}.$$

This quantity is well defined for Lipschitz paths and on $PL_D[0,T]$ the signature is a polynomial expression in $(\gamma_0,\dots,\gamma_{n+1})$. As an example it is straight-forward to compute that for an affine path $\gamma(t) = (1-\frac{t}{T})\gamma_0 + \frac{t}{T}\gamma_1$ the signature is given by

$$\mathbb{X}^n_{[0,T]}(\gamma) = \frac{(\gamma_1-\gamma_0)^{\otimes n}}{n!} := \frac{1}{n!}\underbrace{(\gamma_1-\gamma_0)\otimes(\gamma_1-\gamma_0)}_{n \text{ times}}$$

Another crucial algebraic property of the signature is Chen's relation:

Theorem 4 (Chen's rule)**.** *The signature is multiplicative under concatenation i.e. for $S<U<T$ one has [2]*

$$\mathbb{X}_{[S,T]}(\gamma) = \mathbb{X}_{[S,U]}(\gamma)\otimes\mathbb{X}_{[U,T]}(\gamma) = \sum_{n=0}^{\infty}\sum_{m+l=n}\mathbb{X}^{(m)}_{[S,U]}(\gamma)\otimes X^{(l)}_{[U,T]}(\gamma). \tag{10}$$

13.4 Remarks and open problems

It is interesting to note that is $L=-\frac{1}{2}\Delta$ as we saw in Section 13.1 i.e. when the measures $\mathbb{P}_D$ are the finite dimensional distributions of the Brownian motion law, Theorem 3 recovers the result of Fawcett [3] or Lyons-Victoir [4]:

$$\mathbb{E}(\mathbb{X}(\gamma)) = \exp\left(\frac{1}{2}\sum_{1}^{d} e_i \otimes e_i\right) \qquad (e_i)_{i\in\{1,\dots,d\}} \text{ basis of } \mathbb{R}^d.$$

Some open problems are:

1. The signature (9) is also defined for p rough paths. Can a result similar to 3 hold on the space of rough paths of a given regularity? We have already noticed that μ_D cannot be extended to a unique measure on the Wiener space. However we know that the signature $\mathbb{X}$ is are reparameterization invariant. Possibly μ_D can be extended to the quotient space of parameterization-free rough paths.

2. More weakly, can the measure μ_D on the signature as a subset of the tensor algebra $T(\mathbb{R}^d)$ be extended to a measure on the whole (or possibly truncated) tensor algebra?

3. Since the adopted approach relies on cubature, does it have possible application to numerical methods?

References

[1] Levin, Daniel, and Terry Lyons. *A signed measure on rough paths associated to a PDE of high order: results and conjectures.* Revista Matem??tica Iberoamericana 25.3 (2009): 971-994.

[2] Chen, Kuo-Tsai. *Integration of paths, geometric invariants and a generalized Baker-Hausdorff formula.* Annals of Mathematics (1957): 163-178.

[3] Fawcett, Thomas. *Problems in stochastic analysis: Connections between rough paths and non-commutative harmonic analysis.* Diss. University of Oxford, 2002.

[4] Lyons, Terry, and Nicolas Victoir. *Cubature on Wiener space.* Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences. Vol. 460. No. 2041. The Royal Society, 2004.

Gennady Uraltsev, University of Bonn
email: **guraltse@math.uni-bonn.de**

14 The partial sum process of orthogonal expansion as geometric rough process with Fourier series as an example—an improvement of Menshov-Rademacher theorem

after T. Lyons and D. Yang [2]
A summary written by Dominique Maldague

Abstract

The Menshov-Rademacher theorem gives a sufficient condition for the almost everywhere existence of general orthogonal series in L^2. Under the same condition, the authors prove that the partial sum process of orthogonal series is a geometric 2-rough process. Since being a geometric 2-rough process implies the almost everywhere convergence of the series, this improves the Menshov-Rademacher theorem. The condition can be improved for Fourier series, and an equivalent condition on the limit function is identified. We summarize their results.

14.1 Introduction

The collection $\{u_n\}_{n=0}^{\infty}$ is said to be an orthonormal system in L^2, if there exist measure space $(\Omega, \mathcal{F}, \mu)$ and Hilbert space $(\mathcal{V}, \langle \cdot, \cdot, \rangle)$, such that for each $n \in \mathbb{N}$ $u_n : (\Omega, \mathcal{F}, \mu) \to (\mathcal{V}, \langle \cdot, \cdot, \rangle)$ and

$$\int_\Omega \langle u_n(\omega), u_m(\omega) \rangle \mu(d\omega) = \delta_{mn} \qquad \forall n, m \in \mathbb{N}.$$

Definition 1. *Suppose $\{w(n)\}_{n=0}^{\infty}$ is a sequence of positive non-decreasing numbers. $\{w(n)\}$ is said to be a Weyl multiplier for property p if p holds for all orthogonal series $\sum_{n=0}^{\infty} c_n u_n$ for any orthonormal system $\{u_n\}$ in L^2 and any sequence of numbers $\{c_n\}$ satisfying $\sum_{n=0}^{\infty} w(n)|c_n|^2 < \infty$.*

Menshov [3] and Rademacher [4] independently proved that the exact Weyl multiplier for almost everywhere convergence of general orthogonal series is $(\log_2(n+1))^2$, which is the classical theorem stated below.

Theorem 2. (**Menshov – Rademacher**) *The orthogonal series $\sum_{n=0}^{\infty} c_n u_n$ converges almost everywhere, for any $\{u_n\}_{n=0}^{\infty} \in L^2$ and any sequence of* *numbers* $\{c_n\}_{n=0}^{\infty}$ *satisfying* $\sum_{n=0}^{\infty}(\log_2(n+1))^2|c_n|^2 < \infty$. *Furthermore,* $(\log_2(n+1))^2$ *cannot be replaced by* $o(\log_2(n+1))^2$ *and there exists an absolute constant* C *such that*

$$\int_{\Omega} \max_{0\leq i\leq j<\infty} \|\sum_{n=i}^{j} c_n u_n\|^2 \mu(d\omega) \leq C \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2. \qquad (1)$$

Let X denote the partial sum process of $\sum_{k=0}^{\infty} c_n u_n$, i.e. a process indexed by $\mathbb{N}$ where for each $\omega \in \Omega$ we define

$$X_\omega(n) := \sum_{k=0}^{n} c_k u_k(\omega), \quad \forall n \in \mathbb{N}.$$

In the following, we often write $X(n)$ for $X_\omega(n)$ for ease in notation. The integrand appearing in ((1)) is the ∞-variation squared of the path $X_n(\omega)$ indexed by $\mathbb{N}$. More generally, for any $\alpha : \mathbb{N} \to \mathcal{V}$, we can define the p-variation as

$$\|\alpha\|_{p-var} := \sup_{N\geq 1} \left(\sup_{0\leq k_0<\cdots<k_n\leq N} \sum_{j=0}^{n-1} \|\alpha(k_j+1) - \alpha(k_j)\|^p \right)^{1/p}.$$

For fixed α, the function $p \mapsto \|\alpha\|_{p-var}$ is nonincreasing on $p \in [1,\infty]$, so $\|\alpha\|_{\infty-var} \leq \|\alpha\|_{2-var}$. A. Lewko and M. Lewko [1] strengthened the Menshov-Rademacher theorem by replacing $\|X\|_{\infty-var}$ by $\|X\|_{2-var}$ in (1).

14.2 Geometric 2-rough norm

The authors improved the result of A. Lewko and M. Lewko by strengthening the norm that appears in the integrand in (1). To define this stronger norm, we introduce the following.

For $u, v \in \mathcal{V}$, denote $[u,v] := u \otimes v - v \otimes u$, with $\otimes$ the tensor product. Assume the norm on tensor products satisfies (up to a universal constant)

$$\|u \otimes v\| \leq \|u\|\|v\|, \quad \forall u, v \in \mathcal{V}.$$

Denote $\mathbb{V}^{\otimes 2}$ as the completion of $\{\sum_{i=1}^{n} u_i \otimes v_i | u_i, v_i \in \mathcal{V}, n \geq 1\}$ with respect to the norm selected.

Definition 3. *Suppose $\gamma : \mathbb{N} \to \mathcal{V}$. Define the area of γ, $A(\gamma) : \{(i,j) : i \leq j\} \to \mathcal{V}^{\otimes 2}$ by setting*

$$A(\gamma)(k_1, k_2) = 0, \quad \text{when } k_2 = k_1 \text{ or } k_1 + 1,$$

and when $k_2 \geq k_1 + 2$,

$$A(\gamma)(k_1, k_2) := 2^{-1} \sum_{k_1 \leq j_1 < j_2 \leq k_2 - 1} [\gamma(j_1 + 1) - \gamma(j_1), \gamma(j_2 + 1) - \gamma(j_2)].$$

The area form satisfies what the authors call a multiplicativity property: for $0 \leq k_1 \leq k_2 \leq k_3 < \infty$,

$$\begin{aligned} A(\gamma)(k_1, k_2) = A(\gamma)(k_1, k_2) + A(\gamma)(k_2, k_3) \\ + \frac{1}{2}[\gamma(k_2) - \gamma(k_1), \gamma(k_3) - \gamma(k_2)]. \end{aligned} \tag{2}$$

Definition 4. *If γ is a path indexed by $\mathbb{N}$, and $A(\gamma)$ is the associated area, then γ is a geometric 2-rough path if*

$$\|(\gamma, A(\gamma))\|_{G^{(2)}} := \left(\|\gamma\|_{2-var}^2 + \|A(\gamma)\|_{1-var}\right)^{1/2} < \infty.$$

The motivation for the definition of the geometric 2-rough norm comes from rough path theory: $\|X\|_{G^{(2)}} < \infty$ is a sufficient condition to give meaning to a stochastic differential equation controlled by X (see [5] for details).

14.3 Main results

Theorem 5 (Lyons–Yang '13)**.** *The partial sum process of $\sum_n c_n u_n$, when enhanced by its area process, is a geometric 2-rough process (denotes as $\mathbf{X}$) for any orthonormal system $\{u_n\} \in L^2$ satisfying $\sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2 < \infty$. Moreover, $(\log_2(n+1))^2$ cannot be replaced by $o((\log_2(n+1)))^2$, and*

$$\int_\Omega \|\mathbf{X}_\omega\|_{G^{(2)}}^2 \mu(d\omega) \leq 121 \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2. \tag{3}$$

Note that this is an improvement on the work of A. Lewko and M. Lewko because of the addition of the 1-variation of the area which appears in the geometric 2-rough norm. It is thus important to realize that the area term is not controlled by the 2-variation of the path; see Example 40 in [5] for a sequence of paths with vanishing 2-variation and unbounded area. Also, since area vanishes in the 1-dimension, this is only an improvement in the multidimensional case.

For specific orthonormal systems, the Weyl multiplier for a.e. convergence can be strengthened. As such, we restrict our attention to orthonormal systems which satisfy the following property (which includes Fourier systems).

Definition 6. *$\{u_n\} \in L^2$ is said to have the Hardy property with constant C, if for any sequence of numbers $\{a_n\}_{n=0}^{\infty}$ satisfying $\sum_{n=0}^{\infty} |a_n|^2 < \infty$,*

$$\int_\Omega \sup_{0 \le i \le j < \infty} \left\| \sum_{k=i}^{j} a_k u_k(\omega) \right\| \mu(d\omega) \le C \left(\sum_{n=0}^{\infty} |a_n|^2 \right).$$

Theorem 7. *Suppose $\{u_n\} \in L^2$ has the Hardy property with constant C. Then, for $\{c_n\}$ satisfying $\sum_n \log_2(n+1)|c_n|^2 < \infty$, the partial sum process of $\sum_n c_n u_n$, when enhanced by its area process, is a geometric 2-rough process (denoted as $\mathbf{X}$). Moreover,*

$$\int_\Omega \|\mathbf{X}_\omega\|_{G^{(2)}}^2 \mu(d\omega) \le (604 + 26C) \sum_{n=0}^{\infty} \log_2(n+1)|c_n|^2. \tag{4}$$

14.4 Decomposition into local and long-time behavior

In the proof of Theorem 5, the authors use the same basic technique as is used by A. Lewko and M. Lewko in [1], and as appears in the proof of the Menshov-Rademacher theorem: decompose the partial sum process into a part encoding long range displacement and a part which tracks variation on a prescribed local scale.

Lemma 8. *Suppose $\gamma : \mathbb{N} \to \mathcal{V}$ is a continuous path, and $\{m_n\}_{n=0}^{\infty}$ is a sequence of strictly increasing integers satisfying $\lim_{n\to\infty} m_n = +\infty$. Define $\gamma^1 : \mathbb{N} \to \mathcal{V}$ as $\gamma^1(n) := \gamma(m_n)$, $\forall n \in \mathbb{N}$. Then*

$$\|\gamma\|_{2-var}^2 \le 3 \left(\|\gamma\|_{2-var,[0,m_0]}^2 + \sum_{n=0}^{\infty} \|\gamma\|_{2-var,[m_n,m_{n+1}]}^2 + \|\gamma^1\|_{2-var}^2 \right),$$

and
$$\|A(\gamma)\|_{1-var} \le \|\gamma\|_{2-var}^2 + \|A(\gamma)\|_{1-var,[0,m_0]} + \sum_{n=0}^{\infty} \|A(\gamma)\|_{1-var,[m_n,m_{n+1}]} + \|A^1(\gamma)\|_{1-var}.$$

We control the variation on a finite interval $[m_k, m_{k+1}]$ using the following lemma.

Lemma 9. *Suppose X is the partial sum process of $\sum_{k=0}^{n} c_n u_n$, then,*

$$\int_\Omega \|X_\omega\|^2_{2-var,[0,n]}\mu(d\omega) \le 8(\log_2(n+1))^2 \sum_{k=1}^{n} |c_k|^2. \tag{5}$$

If we define $X_k : \{0, 1, \ldots, m_{k+1} - m_k\} \to \mathcal{V}$ by $X_k(n) = X(m_k + n)$, then we conclude from the lemma that

$$\int_\Omega \|X_k(\omega)\|^2_{2-var,[0,m_{k+1}-m_k]}\mu(d\omega) \le 8(\log_2(m_{k+1} - m_k + 1))^2 \sum_{n=m_k}^{m_{k+1}} |c_n|^2$$

Choosing $m_k = 2^k$ imples that $\log_2(m_{k+1} - m_k + 1) \le \log_2(n+1)$ for $n \in [m_k, m_{k+1}]$, so we obtain the desired upper bound $\sum_{n=m_k+1}^{m_{k+1}}(\log_2(n+1))^2|c_n|^2$. For the area term on $[m_k, m_{k+1}]$, we use the multiplicativity (2) property to translate the local 1-variation of the area to the local 2-variation of the path.

The long-term process X^1 from Lemma 8 with $m_k = 2^k$ is the partial sum process of $\sum_{k=1}^{\infty} a_k v_k(\omega)$ where

$$a_k = \sqrt{\sum_{n=2^k+1}^{2^{k+1}} |c_n|^2} \quad \text{and} \quad v_k = \frac{1}{a_k} \sum_{n=2^k+1}^{2^{k+1}} c_n u_n.$$

Since $\sum_{k=0}^{\infty} k^2|a_k|^2 \le \sum_{k=0}^{\infty} \sum_{n=2^k+1}^{2^{k+1}} (\log_2(n+1))^2|c_n|^2$, it remains to prove (3) with $w(n) = (\log_2(n+1))^2$ replaced by $w(n) = n^2$. One way we exploit this extra room in the upper bound is as follows: (using $\sum_n \frac{1}{w(n)} < \infty$)

$$\begin{aligned}
\|A(X^1)\|_{1-var} &\le \sum_{1\le i<j<\infty} \|a_i v_u\| \|a_j v_j\| = \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} w(i)\frac{\|a_i v_u\|}{w(j)} w(j) \frac{\|a_j v_j\|}{w(i)} \\
&\le \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} \frac{w(i)w(j)}{2} \left(\frac{\|a_i v_u\|^2}{w(j)^2} + \frac{\|a_j v_j\|^2}{w(i)^2} \right) \\
&\le \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \frac{1}{w(j)} \right) w(i) \|a_i v_u\|^2 = K \sum_{i=1}^{\infty} w(i) \|a_i v_i\|^2.
\end{aligned}$$

14.5 Further results

The authors give an equivalent condition for Fourier coefficients to satisfy $\sum_{n=0}^{\infty} \log_2(n+1)|c_n|^2 < \infty$, which involves defining the Sobolev space H^s_{Log}.

In Theorem 7, $\log_2(n+1)$ is not an exact Weyl multiplier for being a geometric 2-rough process. The authors construct an L^2 Fourier series that is a geometric 2-rough process with nontrivial area term, but whose coefficients satisfy $\sum_n w(n)|c_n|^2 = \infty$ for $w(n)$ increasing faster than $(\log_2 \log_2(n+1))^2$. Finally, the authors construct an L^2 Fourier series whose partial sum process has infinite 2-variation almost everywhere.

References

[1] Lewko, A., Lewko, M., Estimates for the square variation of partial sums of Fourier series and their rearrangements, J. Funct. Anal., 262(6), 2561-2607, (2012).

[2] Lyons, T. and Yang, D., The partial sum process of orthogonal expansions as geometric rough process with Fourier series as an example of an improvement of Menshov-Rademacher theorem, J. Funct. Anal. 265 (2013), no. 12, 3067-3103.

[3] Mensov, D., Sur les s??eries de fonctions orthogonales, Fund. Math., 4, 82-105 (1923).

[4] Rademacher, H., Einige S??atze ??uber Reihen von allgemeinen Orthogonal funktionen, Ann. Math., 87, 112-138, (1922).

[5] Yang, D., Notes on Area Operator, Geometric 2-Rough Path and Young integral when $\frac{1}{p} + \frac{1}{q} = 1$, Int. J. Math. Anal., 35(6), 1717-1746, (2012)

Dominique Maldague, University of California, Berkeley
email: **dmaldague@berkeley.edu**

15 Paracontrolled distributions and singular PDEs

after M. Gubinelli, P. Imkeller, and N. Perkowski [1].
A summary written by Immanuel Zachhuber

Abstract

We give a brief introduction of the theory of paracontrolled distributions, which combines the study of the paraproduct, a way of defining multiplication between functions and distributions, and rough path theory(more precisely, the theory of controlled paths), which provides the concept of path-wise defined stochastic integrals and solutions to S(P)DEs.

15.1 Introduction

We want to analyse Stochastic Differential Equations like

$$\begin{aligned} \mathrm{d}u &= F(u)\mathrm{d}B \\ u(0) &= u_0, \end{aligned}$$

where B is a standard Brownian Motion and u_0 is an initial datum. The usual way to approach this kind of problem would be via the Ito integral. Alternatively, one can define a pathwise solution via the theory of Rough Paths if one controls some higher order terms (see [2]). However, we follow a different approach, we reformulate the SDE in a "classical"sense

$$\begin{aligned} \partial_t u &= F(u)\xi \\ u(0) &= u_0, \end{aligned}$$

except that ξ, which is called *white noise*, is only a distribution. More precisely, almost every path of the Brownian Motion has regularity $\mathcal{C}^\alpha$ for $\alpha < 1/2$, the paths of the white noise have regularity $\mathcal{C}^{\alpha-1}$. The Besov-Hölder spaces, $\mathcal{C}^\alpha$, which coincide with the usual Hölder spaces for $\alpha > 0$, will be introduced below. Since one can in general not multiply distributions, we introduce the *paraproduct*, which allows us to multiply distributions with certain Besov-regularity. The standard result for the paraproduct is that one can multiply two distributions $\mu \in \mathcal{C}^\alpha$ and $\nu \in \mathcal{C}^\beta$ if $\alpha + \beta > 1$. Some further analysis is required, since if we assume ξ to be white noise, we can only expect the solution u to have regularity $\alpha < 1/2$, so the sum of the regularities is $2\alpha - 1 < 0$. To overcome this we will need some further mild assumptions. The theory of paracontrolled distributions is an extension of the theory of Rough Paths(more precisely Controlled Paths) in the sense that it allows us to define and analyse path-wise solutions to Stochastic PDEs, for which Ito-Calculus and Rough Path theory are insufficient. An alternative approach to pathwise solutions to (singular) SPDEs is the theory of *Regularity Structures*, see [4].

15.2 Preliminaries

We begin by introducing the concept of the paraproduct, as well as the low-regularity spaces we are interested in, which are a type of Besov space. We briefly introduce some concepts from Littlewood-Paley theory.

Definition 1. Let $\chi, \rho \in C_c^\infty(\mathbb{R}^d)$ be nonnegative radial functions such that

(i) The support of χ is contained in a ball and the support of ρ is contained in an annulus.

(ii) $\chi(z) + \sum_{j=0}^\infty \rho(2^{-j} z) = 1$ for every $z \in \mathbb{R}^d$.

(iii) $\text{supp}(\chi) \cap \text{supp}(\rho(2^{-j}\cdot)) = \emptyset$ for every $j \geq 1$ and $\text{supp}(\rho(2^{-i}\cdot)) \cap \text{supp}(\rho(2^{-j}\cdot)) = \emptyset$ for $|i-j| > 1$

We call this a *dyadic partition of unity.* Furthermore we set

$$\rho_{-1} := \chi \qquad \text{and} \qquad \rho_j := \rho(2^{-j}\cdot) \text{ for } j \geq 0.$$

We now define the Littlewood-Paley as

$$\Delta_{-1} := \mathcal{F}^{-1}\chi\mathcal{F} \qquad \text{and} \qquad \Delta_j := \mathcal{F}^{-1}\rho_j\mathcal{F} \text{ for } j \geq 0.$$

Then, for any tempered distribution $u \in \mathcal{S}'$, we get $\Delta_j u = K_j * u$ where $K_j = \mathcal{F}^{-1}\rho_j$ for all $j \geq -1$, $\sum_{j=-1}^\infty \Delta_j u = u$ in $\mathcal{S}'$ and in particular $\Delta_j u$ is a smooth function for every $j \geq -1$.
Lastly we introduce the Besov spaces

$$B_{p,q}^\alpha(\mathbb{R}^d; \mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d; \mathbb{R}^n) : ||u||_{B_{p,q}^\alpha} := \left(\sum_{j=-1}^\infty (2^{j\alpha}||\Delta_j u||_{L^p})^q \right)^{1/q} < \infty \right\}$$

for $\alpha \in \mathbb{R}, p, q \in [1, \infty]$ with the obvious modification for $q = \infty$. We call $\mathcal{C}^\alpha := B^\alpha_{\infty,\infty}$ the Besov-Hölder spaces. Analogously we can define everything on the $d-$dimensional torus, $\mathbb{T}^d$.

The proof of existence of a dyadic partition of unity can for example be found in [3]. Next we state a simple but useful regularity result.

Lemma 2. *(i) Let $\mathcal{A}$ be an annulus, let $\alpha \in \mathbb{R}$ and let u_j be a sequence of smooth functions, such that $\mathcal{F}u_j$ have support in $2^j\mathcal{A}$ and which satisfy $||u_j||_\infty \lesssim 2^{-j\alpha}$ for all j. Then*

$$u = \sum_{j \geq -1} u_j \in \mathcal{C}^\alpha \quad \textit{and} \quad ||u||_\alpha \lesssim \sup_{j \geq -1} \{2^{j\alpha} ||u_j||_\infty\}.$$

(ii) Let $\mathcal{B}$ be a ball, let $\alpha > 0$ and let u_j be a sequence of smooth functions, such that $\mathcal{F}u_j$ have support in $2^j\mathcal{B}$ and which satisfy $||u_j||_\infty \lesssim 2^{-j\alpha}$ for all j. Then

$$u = \sum_{j \geq -1} u_j \in \mathcal{C}^\alpha \quad \textit{and} \quad ||u||_\alpha \lesssim \sup_{j \geq -1} \{2^{j\alpha} ||u_j||_\infty\}.$$

15.3 Paracontrolled calculus

15.3.1 The paraproduct

In terms of our Littlewood-Paley decomposition we can (formally) decompose the product of two distributions f and g

$$fg = \sum_{j \geq -1} \sum_{i \geq -1} \Delta_i f \Delta_j g =: f \prec g + f \succ g + f \circ g,$$

where we define

$$f \prec g = g \succ f := \sum_{j \geq -1} \sum_{i=-1}^{j-2} \Delta_i f \Delta_j g \quad \text{and} \quad f \circ g := \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g$$

and we call $f \prec g$ and $f \succ g$ *paraproducts*, which are the off-diagonal summands, and $f \circ g$, which are the diagonal terms in the sum, the *resonant*

part.
Furthermore we introduce the notations

$$f \preccurlyeq g := f \prec g + f \circ g \;\text{ and }\; S_j f := \sum_{i=-1}^{j-1} \Delta_i f.$$

A crucial observation is that the Fourier transform of each summand in the paraproduct $f \prec g$ has support in a suitable annulus and the Fourier transform of each summand in $f \circ g$ has support in a suitable ball.
We state some simple bilinear estimates for the paraproducts, which are essentially applications of Lemma 2.

Lemma 3. *For any $\beta \in \mathbb{R}$ the following holds*

(i) $||f \prec g||_\beta \lesssim_\beta ||f||_\infty ||g||_\beta$

(ii) if in addition $\alpha < 0$,
$||f \prec g||_{\alpha+\beta} \lesssim_{\alpha,\beta} ||f||_\alpha ||g||_\beta.$

(iii) For $\alpha + \beta > 0$ we have
$||f \circ g||_{\alpha+\beta} \lesssim_{\alpha,\beta} ||f||_\alpha ||g||_\beta$

15.3.2 Paracontrolled distributions and RDEs

Consider a Rough Differential Equation(RDE) of the form

$$\partial_t u = F(u)\xi \qquad\qquad u(0) = u_0, \tag{1}$$

where $u_0 \in \mathbb{R}^d$, $u : \mathbb{R} \to \mathbb{R}^d$ is a continuous vector-valued function, $\xi : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued distribution with values in $\mathcal{C}^{\alpha-1}$ for some $\alpha \in (1/3, 1)$ and $F : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^d)$ is a family of vector fields. A natural approach is to see this as the limit of classical ODEs

$$\partial_t u^\varepsilon = F(u^\varepsilon)\xi^\varepsilon \qquad\qquad u^\varepsilon(0) = u_0, \tag{2}$$

where the ξ^ε denotes some smooth approximation such that $\xi^\varepsilon \to \xi$ in $\mathcal{C}^{\alpha-1}$.
In order to pass to the limit, we need some a priori estimates for the solution, which only depend on the $\alpha - 1$ norm of ξ^ε. For ease of notation we will drop the ε even though we will work with the regularised solutions.
We expect u to have $\mathcal{C}^\alpha$ regularity, since it should gain one derivative compared with $F(u)\xi$, which can not be more regular than ξ, so $\mathcal{C}^{\alpha-1}$. Moreover,

assuming F is sufficiently smooth, $F(u)$ will also have regularity $\mathcal{C}^\alpha$. Thus we can decompose the right hand side of (1) in three terms with the folllowing regularities(using Lemma 3)

$$\underbrace{F(u) \prec \xi}_{\alpha-1} + \underbrace{F(u) \circ \xi}_{2\alpha-1} + \underbrace{F(u) \succ \xi}_{2\alpha-1}. \tag{3}$$

Note, however, that unless $2\alpha - 1 > 0$, we can not bound the resonant term $F(u) \circ \xi$ by the α norm of u and the $\alpha - 1$ norm of ξ. If $F \in C^2$ we can use a paralinearisation result(Lemma 5) to rewrite this term as

$$F(u) \circ \xi = F'(u)(u \circ \xi) + \Pi_F(u, \xi), \tag{4}$$

where the remainder term $\Pi_F(u, \xi)$ is well-defined as long as $3\alpha > 1$, $u \in \mathcal{C}^\alpha$ and $\xi \in \mathcal{C}^{\alpha-1}$. In this case it belongs to $\mathcal{C}^{3\alpha-1}$. The problem now is to make sense of the resonant term $u \circ \xi$. As a next step, we want to use the fact that the solution u has a specific structure, thus we *postulate* that u is given by the following *paracontrolled ansatz*:

$$u = u^\vartheta \prec \vartheta + u^\sharp,$$

where $u^\vartheta, \theta \in \mathcal{C}^\alpha$ and the remainder $u^\sharp$ is in $\mathcal{C}^{2\alpha}$. Note the similarity to the theory of Controlled Paths, see [2]. Inserting this ansatz, we obtain

$$u \circ \xi = (u^\vartheta \prec \vartheta) \circ \xi + u^\sharp \circ \xi = u^\vartheta(\vartheta \circ \xi) + C(u^\vartheta, \vartheta, \xi) + u^\sharp \circ \xi, \tag{5}$$

where the commutator is defined as $C(u^\vartheta, \vartheta, \xi) := (u^\vartheta \prec \vartheta) \circ \xi - u^\vartheta(\vartheta \circ \xi)$ and has $\mathcal{C}^{3\alpha-1}$ regularity under the above assumptions, see Lemma 4 below. Note that the term $u^\sharp \circ \xi$ causes no further problems, since it is bounded in $\mathcal{C}^{3\alpha-1}$. The only remaining problematic term is $\vartheta \circ \xi$. For this term we have to *assume* that it has regularity $\mathcal{C}^{2\alpha-1}$. Note again the parallel to the theory of Controlled Paths, where we need control of the second order "iterated integrals", see [2]; see section 3.4 of [1] to see the equivalence of the two.
With all this in hand, we find that the right-hand side of (1) is well-defined and a continuous function of $(u, u^\vartheta, u^\sharp, \vartheta, \xi, \vartheta \circ \xi)$.
If we now insert the paracontrolled ansatz into (1) we obtain

$$\partial_t u = \partial_t(u^\vartheta \prec \vartheta + u^\sharp) = \partial_t u^\vartheta \prec \vartheta + u^\vartheta \prec \partial_t \vartheta + \partial_t u^\sharp \tag{6}$$

if we moreover choose ϑ s.t. $\partial_t \vartheta = \xi$ and set $u^\vartheta = F(u)$ we get the following equation for the remainder $u^\sharp$

$$\partial_t u^\sharp = F'(u)F(u)(\vartheta \circ \xi) + F(u) \succ \xi - (\partial_t F(u) \prec \vartheta) + \tag{7}$$
$$+ F'(u)F'(u)C(F(u), \vartheta, \xi) + F'(u)(u^\sharp \circ \xi) + \Pi_F(u, \xi). \tag{8}$$

This, together with the equation $u = F(u) \prec \vartheta + u^\sharp$, allows us to establish an a priori estimate for the solution u in terms of the data $(u_0, ||\xi||_{\alpha-1}, ||\vartheta \circ \xi||_{2\alpha-1})$. We state the full result below in Theorem 6.

15.3.3 Commutator estimates and paralinearisation

Lemma 4. *Let $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}$ s.t. $\alpha + \beta + \gamma > 0$ but $\beta + \gamma < 0$. Then for smooth f, g, h the trilinear operator*

$$C(f, g, h) = ((f \prec g) \circ h) - f(g \circ h)$$

is bounded in the following sense

$$||C(f, g, h)||_{\alpha+\beta+\gamma} \lesssim ||f||_\alpha ||g||_\beta ||h||_\gamma$$

Thus C can be uniquely extended to a bounded trilinear operator in $\mathcal{L}^3(\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma; \mathcal{C}^{\alpha+\beta+\gamma})$.

Lemma 5. *Let $\alpha \in (0,1)$, $\beta \in (0, \alpha]$, and $\gamma \in \mathbb{R}$ s.t. $\alpha + \beta + \gamma > 0$ but $\alpha + \gamma < 0$. Let $F \in C_b^{1+\beta/\alpha}$. There exists a locally bounded map $\Pi_F : \mathcal{C}^\alpha \times \mathcal{C}^\gamma \to \mathcal{C}^{\alpha+\beta+\gamma}$ such that*

$$F(f) \circ g = F'(u)(f \circ g) + \Pi_F(f, g)$$

for all $f \in \mathcal{C}^\alpha$ and smooth g.

15.4 Rough Differential equations

We state the final result for RDEs. Strictly speaking we need some slightly technical localisation results, but we will gloss over them here.

Theorem 6. *Let $\alpha > 1/3$. Assume $(\xi^\varepsilon)_\varepsilon$ is a family of smooth functions with values in $\mathbb{R}^n$, $(u_0^\varepsilon)_\varepsilon$ is a family of initial conditions in $\mathbb{R}^d$ and F is a family of C_b^3 vector fields. Suppose there exist limits $(u_0, \xi, \vartheta, \eta)$ such that*

$(u_0^\varepsilon, \xi^\varepsilon, \vartheta^\varepsilon, (\vartheta^\varepsilon \circ \xi^\varepsilon)) \to (u_0, \xi, \vartheta, \eta)$ in $\mathbb{R} \times \mathcal{C}^{\alpha-1} \times \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha-1}$, where ϑ^ε and ϑ are solutions to $\partial_t \vartheta^\varepsilon = \xi^\varepsilon$ and $\partial_t \vartheta = \xi$ respectively. Let for $\varepsilon > 0$, the function u^ε be the unique global solution to

$$\partial_t u^\varepsilon = F(u^\varepsilon)\xi^\varepsilon \qquad\qquad u^\varepsilon(0) = u_0^\varepsilon. \tag{9}$$

Then there exists $u \in \mathcal{C}^\alpha_{\text{loc}}$ s.t. $u^\varepsilon \to u$ in $\mathcal{C}^\alpha_{\text{loc}}$ as $\varepsilon \to 0$. The limit depends on the data $(u_0, \xi, \vartheta, \eta)$ but not on the approximation family.

References

[1] Gubinelli, Massimiliano, Peter Imkeller, and Nicolas Perkowski. *Paracontrolled distributions and singular PDEs.* Forum of Mathematics, Pi. Vol. 3. Cambridge University Press, 2015.

[2] Gubinelli, Massimiliano. “Controlling rough paths”. arXiv preprint math/0306433 (2003).

[3] Bahouri, Hajer, Jean-Yves Chemin, and Rapha??l Danchin. Fourier analysis and nonlinear partial differential equations. Vol. 343. Springer Science Business Media, 2011.

[4] Hairer, Martin. ”A theory of regularity structures.” Inventiones mathematicae 198.2 (2014): 269-504.

Immanuel Zachhuber, University of Bonn
email: **s6imzach@uni-bonn.de**

16 The Nash-Moser theorem and paradifferential operators

after L. Hörmander [2]
A summary written by Polona Durcik

Abstract

The theorems of Nash-Moser type play an important role in the theory of non-linear PDEs. The paper in question discusses a related approach for proving existence theorems for some non-linear PDEs. It replaces the Nash-Moser technique by non-linear functional analysis combined with the paradifferential calculus.

16.1 Introduction

The paper [2] consists of two parts. In the first part, paradifferential operators in the spirit of Bony [1] are used to prove a version of the isometric embedding theorem. The isometric embedding theorem is due to Nash [3]. In the second part of the paper, this approach is put into an abstract setting of the same generality as the first versions of Nash-Moser techniques.

16.2 The isometric embedding problem

Let M be a compact smooth Riemannian manifold of dimension n. The question is to find for some large N an isometric embedding

$$u : M \to \mathbb{R}^N.$$

This problem was first solved by Nash [3], who reduced it to the following perturbation problem.

By a suitable embedding of M into a torus we may assume that M is the torus $(\mathbb{R}/\mathbb{Z})^n$, which allows us to use global coordinates on M. Let u_0 be a smooth embedding of M into $\mathbb{R}^N$ which is *free*, that is, such that the derivatives

$$\partial u_0/\partial x_j, \quad \partial^2 u_0/\partial x_j \partial x_k \quad 1 \leq j \leq k \leq n,$$

are linearly independent. Such an embedding exists provided N is large enough. The problem of finding an isometric embedding of M is then reduced to showing that if ϱ is large enough and $g \in C_*^{\varrho+1}$ is given, periodic with values in $\mathbb{R}^{\binom{n+1}{2}}$, one finds $u \in C_*^{\varrho+1}$ periodic with values in $\mathbb{R}^n$ such that

$$\Phi(u+u_0) = \Phi(u_0) + g, \tag{1}$$

where $\Phi(v)$ is the pullback $|dv|^2$ of the Euclidean metric in $\mathbb{R}^N$ by v. We say that u belongs to $C_*^\varrho = C_*^\varrho(\mathbb{R}^n)$, $\varrho > 0$, if

$$\sup_{k \geq -1} 2^{k\varrho} \|u_k\|_{\mathrm{L}^\infty} < \infty, \tag{2}$$

where u_k are defined on the Fourier side by $\widehat{u_{-1}} := \varphi\widehat{u}$, $\widehat{u_k} := \psi(2^{-k}\cdot)\widehat{u}$ if $k \geq 0$, and $\varphi \in C_0^\infty(\mathbb{R}^n)$ is supported in the unit ball and is equal to 1 in the concentric ball with radius 1/2, $\psi(\xi) = \varphi(2^{-1}\xi) - \varphi(\xi)$. We use (2) to define the C_*^ϱ-norm. If ϱ is not an integer, C_*^ϱ is the Hölder class C^ϱ (the class of $\lfloor\varrho\rfloor$-times continuously differentiable functions u which are bounded and their derivatives up to order $\lfloor\varrho\rfloor$ are bounded, and for $|\sigma| = \lfloor\varrho\rfloor$, $D^\sigma u$ is Hölder continuous with exponent $\varrho - \lfloor\varrho\rfloor$). If ϱ is an integer, C_*^ϱ is strictly larger than the class of ϱ-times continuously differentiable functions, bounded with bounded derivatives.

We return to the equation (1). Let φ, ψ be as above. Define the function

$$\chi(\xi,\eta) := \sum_{k=1}^{\infty} \varphi(\xi 2^{2-k})\psi(\eta 2^{-k})$$

and write $a_\chi(x,\eta)$ for the inverse Fourier transfor of $\widehat{a}(\xi)\chi(\xi,\eta)$ with respect to ξ. For $a \in C_*^\varrho$ define the following *paramultiplication by* a:

$$T_a u(x) = \int e^{-2\pi i x\cdot\eta} a_\chi(x,\eta)\widehat{u}(\eta)d\eta, \quad u \in \mathcal{S}.$$

Then T_a is continuous from C_*^σ to C_*^σ for every σ and depends linearly on a. Note that T_a depends on the choice of χ. However, the difference between operators with different choices of χ improves differentiability and it is negligible in applications.

Now, since u_0 is free, one can easily find a right inverse of the Frechet differential $\Phi'(u)$, which we denote by $\Psi(u)$. It is a multiplication by a matrix with non-linear second order differential operators as coefficients. Then we consider the equation

$$u - \Psi(u_0)h - T_{\Psi(u+u_0)-\Psi(u_0)}h = 0, \tag{3}$$

where $h \in C_*^{\varrho+1}$ is given with values in $\mathbb{R}^{\binom{n+1}{2}}$, $\varrho > 1$. The equation has a unique solution $u = U(h)$ which is a C^∞ function from $C_*^{\varrho+1}$ to $C_*^{\varrho+1}$ in a neighbourhood of 0, with $U(0) = 0$ and differential $\Psi(u_0)h$ at 0.

There are two main reasons for introducing the operator T_a. The first one is that one can "paralinearize" the equation (1). That is, one can write

$$\Phi(u + u_0) = \Phi(u_0) + T_{\Phi'(u+u_0)}u + R(u), \tag{4}$$

where $R(u)$ is given by

$$R(u) = \Phi(u + u_0) - \Phi(u_0) - T_{\Phi'(u+u_0)}u.$$

The map $u \mapsto R(u)$ is infinitely Frechet differentiable. More precisely, for $\varrho > 0$ the map $u \mapsto R(u)$ belongs to $C^\infty(V, C_*^{2\varrho})$ where V is a neighbourhood of 0 in $C_*^{\varrho+1}(\mathbb{R}^n, \mathbb{R}^N)$.

Second, the fact that

$$\Phi'(u + u_0)\Psi(u + u_0) = \text{identity}$$

leads to the fact that

$$T_{\Phi'(u+u_0)}T_{\Psi(u+u_0)} - \text{identity} \tag{5}$$

is an infinitely differentiable function of $u \in C_*^{\varrho+1}$, which takes values in $\mathcal{L}(C_*^{\sigma+1}, C_*^{\sigma+\varrho-1})$ for arbitrary σ if $\varrho > 2$. (For the precise definition of $T_{\Phi'(u+u_0)}$ see [2]). One can "parainvert" $T_{\Phi'(u+u_0)}$ by the operator $T_{\Psi(u+u_0)}$.

Combining (3), (4) and (5) allows us to solve (1) locally using the implicit function theorem, provided $\varrho > 2$. This in turn yields that for the torus M one has

Theorem 1. *If $\varrho > 3$ and $g \in C_*^\varrho(M, \mathbb{R}^{\binom{n+1}{2}})$ is sufficiently small, then one can find $u \in C_*^\varrho(M, \mathbb{R}^N)$ so small that the map $u + u_0 : M \to \mathbb{R}^N$ is an isometric embedding for the metric $g + g_0$.*

Remark: Nash [3] obtained an embedding theorem for the classical C^ϱ spaces where $\varrho \geq 3$ is an integer.

16.3 An abstract Nash-Moser theorem

In this section we put the method from Section 16.2 into a more general setting. The method is applicable to general non-linear differential equations for which there exists a non-linear differential operator which is a right inverse of the differential (such as $\Psi(u)$ in the previous section).

The first step is to define an abstract version of the spaces C_*^ϱ. Let E^a, $a \geq 0$ be a decreasing family of Banach spaces with injections $E^b \hookrightarrow E^a$ of norm ≤ 1 when $b \geq a$. Assume that we are given continuous linear operators

$$S(\theta) : E^0 \to E^\infty = \bigcap_1^\infty E^a, \quad \theta \geq 1,$$

such that with constants bounded when a and b are bounded one has

$$\begin{aligned}
&\|S(\theta)u\|_b \leq C\|u\|_a, \quad b \leq a \\
&\|S'(\theta)u\|_b \leq C\theta^{b-a-1}\|u\|_a \\
&\text{if } u \in E^0 \text{ and } S(\theta)u \to v \text{ in } E^0 \text{ as } \theta \to \infty, \text{ then } u = v.
\end{aligned}$$

For $a > 0$ we define E_*^a to be the set of all $u \in E^0$ such that for some M

$$\|u\|_0 \leq M; \quad \|S'(\theta)u\|_0 \leq M\theta^{-a-1}, \quad \|S'(\theta)u\|_{a+1} \leq M, \quad \theta \geq 1, \tag{6}$$

and we define $\|u\|_a^*$ as the smallest M which can be used in (6).

Now we consider two scales of Banach spaces $\{E^a\}$ and $\{F^b\}$ with these properties and write $S_E(\theta)$ and $S_F(\theta)$ for the corresponding smoothing operators $S(\theta)$. Let V be a neighbourhood of 0 in some E_*^μ and let $\Phi : E^\infty \cap V \to F^\infty$. For a continuous extension of Φ and some $\alpha > \mu$ we want to find a solution $u \in E_*^\alpha$ of

$$\Phi(u) = f. \tag{7}$$

We assume that Φ has a differential $\Phi'(u)$ when $u \in E^\infty \cap V$ and that $\Phi'(u)$ has a right inverse $\Psi(u)$ such that for certain a_1, a_2 with $0 \leq a_1 < \alpha < a_2$

$$\|\Psi(u)g\|_a \leq C \sum (1 + \|u\|_{A_j(a)})\|g\|_{B_j(a)}, \tag{8}$$

whenever $a_1 \leq a \leq a_2$, $u \in E^\infty \cap V$, $g \in F^\infty$, the sum is finite, and $A_j(a)$, $B_j(a)$ are increasing linear functions. We also assume that $(E^\infty \cap V) \times F^\infty \ni (u, g) \mapsto \Psi(u)g \in E^{a_2}$ is continuous.

For instance, with Ψ as in the previous section and with Hölder norms and $\mu > 2$, one has for $a > \mu$

$$\|\Psi(u)g\|_a \leq C((1 + \|u\|_{a+2})\|g\|_0 + \|g\|_a).$$

The first step in finding solutions to (7) is to defined a "parainverse" corresponding to $\Psi(u)$. Let V_1 be a neighbourhood of 0 in E^μ_* such that $S_E(\theta)V_1 \subset V$ if $\theta \geq 1$. Let $\alpha > \mu$ and $W \subset V_1$ a neighbourhood of 0 in E^α_*. Assume that for some $\beta > 0$

$$\max(a_1, \mu) < \alpha < a_2, \quad B_j(a) \leq \beta - \alpha + a, \quad A_j(a) + B_j(a) < a + \beta, \tag{9}$$

where $a_1 \leq a \leq a_2$. If $u \in W$, $g \in F^\beta_*$, then

$$T_\Psi(u)g := \int_1^\infty \Psi(S_E(t)u)S'_F(t)g\,dt \quad \in E^\alpha_*. \tag{10}$$

The next step is to consider the equation analogous to (3)

$$u - \Psi(0)g - (T_\Psi(u) - T_\Psi(0))g = 0, \tag{11}$$

where $g \in F^\beta_*$, and we look for $u \in E^\alpha_*$.

Theorem 2. *Assume that $\Psi(u)$ satisfies (8) for a neighbourhood V of $0 \in E^\mu_*$ and that (9) holds. Assume also that $E^\infty \times F^\infty \ni (u, g) \mapsto \Psi(u)g \in E^{a_2}$ is continuous when $u \in V$ and that the scale $\{E^a\}$ is compact, i.e. that $E^a \hookrightarrow E^b$ is compact for all $a > b$. Then $T_\Psi(u)$ can be defined and (10) holds for some neighbourhood if W of 0 in E^α_*. The equation (11) has a solution $u \in W$ for all $g \in F^\beta_*$, with sufficiently small norm.*

The proof Theorem 2 is via the Leray-Schauder fixed point theorem. For $g \in F^\beta_*$ we now set

$$\widetilde{u}(t) := \Psi(0)S_F(1)g + \int_1^t \Psi(S_E(s)u)S'_F(s)g\,ds. \tag{12}$$

Now we are ready to state the main result of [2].

Theorem 3. *Let V be a neighbourhood of 0 in E^μ_* and assume that Φ is differentiable from $V \cap E^{a_2}$ to F^β, $\Phi(0) = 0$. Assume that the differential satisfies a Lipschitz condition*

$$\|(\Phi'(u) - \Phi'(v))w\|_\beta \leq C \sum (1 + \|u\|_{m'_j} + \|v\|_{m'_j})\|u - v\|_{m''_j}\|w\|_{m'''_j}$$

for all $u, v \in E^a \cap V$, $w \in E^\infty$, where m'_j, m''_j, $m'''_j \in [a_1, a_2]$, and that $\alpha > \max(m''_j + m''_j)/2$ and $\alpha > \max(m''_j + m''_j + m'''_j)/3$ for some $\alpha > 0$. Assume also that the differential has an inverse $\Psi(u)$ satisfying the hypotheses of Theorem 2. Let the scale $\{E^a\}$ of Banach spaces be compact. Then one can for every $f \in F^\beta_*$ with sufficiently small norm find $u \in E^\alpha_*$ with small norm such that $\widetilde{u}(t)$ defined by (12) converges to u and $\Phi(\widetilde{u}(t)) \to f$ as $t \to \infty$. The convergence is strong in E^a resp. F^b when $a < \alpha$ and $b < \beta$ respectively.

References

[1] Bony, J.-M., *Paracomposition et operateurs pradifferentiels*, Comm. Partial Diff. Equations **11** (1986), 87-121.

[2] Hörmander, L., *The Nash-Moser theorem and paradifferential operators*, Analysis, et cetera, 429-449, Academic Press, Boston, MA, 1990.

[3] Nash, J., *The embedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20-63.

POLONA DURCIK, UNIVERSITY OF BONN
email: **durcik@math.uni-bonn.de**

17 Geometric vs non-geometric rough paths

after Martin Hairer and David Kelly [1]
A summary written by Nikolay Barashkov

17.1 Introduction

Recall the definition of a geometric rough path: A geometric rough path takes values in $T(V)^*$ where

$$T(V) = \bigoplus_{n=0}^{\infty} \bigotimes_{i=1}^{n} V$$

The interpretation is that we assign the values of the iterated integrals by

$$\langle \mathbb{X}_t, e_{i_1 \dots i_n} \rangle = \int_0^t \dots \int_0^{r_2} dX_{r_1}^{i_1} \dots dX_{r_n}^{i_n} \ ,$$

For geometric rough paths we assume that we can integrate by parts which yields the identity

$$\langle \mathbb{X}_t, e_{i_1 \dots i_n} \rangle \langle \mathbb{X}_t, e_{j_1 \dots j_m} \rangle = \langle \mathbb{X}_t, e_{i_1 \dots i_n} \sqcup\!\sqcup e_{j_1 \dots j_m} \rangle$$

Where $\sqcup\!\sqcup$ denotes the shuffle product. Using this we can express Integrals like

$$\begin{aligned} \int_s^t \left(\int_s^{v_3} dX_{v_1}^k \right) \left(\int_s^{v_3} dX_{v_2}^j \right) dX_{v_3}^i &= \int_s^t \langle \ \mathbb{X}_{st}, e_k \rangle \langle \mathbb{X}_{st}, e_j \rangle dX_{v_3}^i \\ &= \int_s^t \langle \mathbb{X}_{st}, e_i \sqcup\!\sqcup e_j \rangle dX_{v_3}^i \\ &= \langle \mathbb{X}_{st}, e_{kji} \rangle + \langle \mathbb{X}_{st}, e_{jki} \rangle \end{aligned}$$

But what happens if we do not have partial integration at our disposal, like in Ito calculus? Then we must define objects like the above separately. To do this we must replace T(V) by a bigger space and give it a structure similar to that of the tensor algebra. That structure will be the one of a Hopf algebra.

17.2 Hopf Algebras

A Hopf algebra is a special kind of bialgebra, so we will first define the latter. A bialgebra arises naturally when one algebra is in some sense *acting* on another. To this end, let H be a vector space and let H^* be another vector space, acting linearly on h via the pairing $\langle \cdot, \cdot \rangle : H^* \otimes H \to \mathbb{R}$. Suppose moreover that H is actually an algebra, with some product $\cdot : H \otimes H \to H$ and unit element 1. In many natural situations, the space H^* is also an algebra, with some other product $\star : H^* \otimes H^* \to H^*$ and a 1^*, which acts as the dual element of 1.

It is often advantageous to superimpose the structure from H^* onto H, so that we simply have a vector space H^* acting on a more structured space H. To be precise, the product $\star$ can be encoded into H by a map $\Delta : H \to H \otimes H$ called a coproduct. The coproduct is the dual of $\star$ in the sense that

$$\langle f \star g, h \rangle = \langle f \otimes g, \Delta h \rangle \ ,$$

for every $f, g \in H^*$ and $h \in H$. The triple $(H, \cdot, \Delta)$ is then called a bialgebra , provided certain consistency relations between the product and coproduct are satisfied.

Suppose that some $f \in H^*$ has an inverse $f^{-1} \in H^*$, satisfying $f \star f^{-1} = f^{-1} \star f = 1^*$ Since we want all the structure of H^* to be contained in H, we must encode an inverse map into H. In fact, we introduce a map $S : H \to H$ such that $S^* : H^* \to H^*$ is the inverse map, satisfying $S^* f \star f = f \star S^* f = 1^*$. The map S is called the *antipode*. But since we only want to work on H and not H^*, the dual requirement for S is that

$$(Id \otimes S)\Delta h = (S \otimes Id)\Delta h = \langle 1^*, h \rangle 1$$

An important example of Hopf Algebras is the tensor algebra: let $e_a = e_{a_1} \otimes \cdots \otimes e_{a_n}$ and $e_b = e_{b_1} \otimes \cdots \otimes e_{b_m}$. T(V) can be made into a Hopf algebra by with the shuffle product defined as

$$e_a \sqcup\!\sqcup e_b = \sum_{c \in shuf(a,b)} e_c$$

where $c \in shuf(a, b)$ iff c is a permutation of the index sequence $(a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ which preserves the original ordering of the index sequences a and b respectively. For example $shuf(ij, k) = \{kij, ijk, ikj\}$ but

jik is not int the set as the original ordering of i and j is not preserved. The coproduct is given as

$$\Delta e_c = \sum_{(a,b)=c} e_a \otimes e_b$$

This gives a bialgebra structure and one can prove that an antipode can be found to make this into an Hopf Algebra.
For branched rough paths we will replace T(V) by the following object: Let $\mathcal{T}$ be the set of rooted labeled trees. The Connes-Kreimer Hopf Algebra $\mathcal{H}$ is the commutative polynomial algebra generated by $\mathcal{T}$. The product is the usual polynomial product and the coproduct will be given as follows: We will say that the pair $(\tau_1 \dots \tau_m) \otimes \tau_0$ is an admissible cut of $\tau \in \mathcal{T}$, if one can obtain τ by *attaching* the trees $\tau_1, \dots, \tau_m$ to the nodes of τ_0. Then

$$\Delta \tau = \sum_{(\tau)} \tau^{(1)} \otimes \tau^{(2)}$$

where we sum over all admissible cuts. Then we extend the coproduct to the whole of $\mathcal{H}$ by requiring it to multiplicative, namely

$$\Delta(\tau_1 \dots \tau_n) = \Delta\tau_1 \dots \Delta\tau_n$$

the product on the right hand side is the tensor product of the polynomial multiplication with itself. Again it can be proven that we can find an antipode to turn $\mathcal{H}$ into a Hopf Algebra. For $g \in \mathcal{H}^*$ we say $g \in G(\mathcal{H})$ if

$$\langle g, h_1 h_2 \rangle = \langle g, h_1 \rangle \langle g, h_2 \rangle$$

Proposition 1. *The pair $(G(\mathcal{H}), \star)$ is a group with inverses given by $g^{-1} = S^* g$, where S^* is the adjoint of the antipode*

We define the truncated group-like elements $G_N(\mathcal{H})$, obtained from $G(\mathcal{H})$ by quotienting out the ideal

$$\bigoplus_{k=N+1}^{\infty} \mathcal{H}^*_{(k)}$$

Where $h \in \mathcal{H}_{(k)}$ if $h = \tau_1 \dots \tau_n$ with $|\tau_1| + \dots + |\tau_n| = k$ if $|\tau| =$ number of vertices in τ
Now we are finally ready to define branched rough paths.

17.3 Branched Rough Paths

Definition 2. *A map $\mathbb{X} : [0,T] \to G_N(\mathcal{H})$ is called a γ-Hölder* branched rough path *if it satisfies*

$$\sup_{s \neq t} \frac{|\langle \mathbb{X}_{st}, \tau \rangle|}{|t-s|^{\gamma|\tau|}} < \infty ,$$

for every $\tau \in \mathcal{H}$ and where $\mathbb{X}_{st} = \mathbb{X}_s^{-1} \star \mathbb{X}_t$. If $\langle \mathbb{X}_{st}, \bullet_i \rangle = \delta X^i_{st}$ for each $i = 1 \ldots d$, then we call $\mathbb{X}$ a branched rough path above X.

It is a perhaps surprising result that we can "embed" branched rough paths into geometric rough paths, more precisely, let $\mathcal{B}_N$ be the vectorspace generated by $\mathcal{T}_N$, the set of trees with at most N vertices.

Theorem 3. *Let $X = (X^i)_{i=1\ldots d}$ be a path in $\mathbb{R}^d$ and $\mathbb{X}$ a γ-Hölder branched rough path in $\mathcal{H}$ such that $\langle \mathbb{X}_{st}, \bullet_i \rangle = \delta X^i_{st}$. Then there exists*

1. *a path $\bar{X} = (\bar{X}^\tau)_{\tau \in \mathcal{T}_N}$ taking values in $\mathcal{B}_N$, with $\pi_{\mathcal{B}_1}(\bar{X}) = X$,*
2. *a γ-Hölder geometric rough path $\bar{\mathbb{X}}$ in $T^{(N)}(\mathcal{B}_N)$ satisfying $\langle \bar{\mathbb{X}}_{st}, \tau \rangle = \delta \bar{x}^\tau_{st}$ for each $\tau \in \mathcal{T}$ and*
3. *a graded morphism of Hopf algebras $\psi : \mathcal{H} \to T(\mathcal{B}_N)$,*

such that

$$\langle \mathbb{X}_{st}, h \rangle = \langle \bar{\mathbb{X}}_{st}, \psi(h) \rangle,$$

for every $h \in \mathcal{H}$.

The geometric path provided by 3 is highly non canonical and has a great deal of redundancy . We will now provide a rough outline of the proof of 3. First we construct ψ with the following lemma:

Lemma 4. *There exists a graded morphism of Hopf algebras $\psi : (\mathcal{H}, \cdot, \Delta) \to (T(\mathcal{B}), ш, \bar{\Delta})$ satisfying*

$$\psi(\tau) = \tau + \psi_{n-1}(\tau) ,$$

for any $\tau \in \mathcal{T}_n$, where ψ_{n-1} denotes the projection of ψ onto $T(\mathcal{B}_{n-1})$.

The proof of the lemma is by induction and computation. After having found ψ we construct our path. This happens step by step. First we define

$$\hat{\mathbb{X}}_t^{(1)} = \exp\left(\sum_{i=1}^{d} \langle \mathbb{X}_t, \bullet_i \rangle \bullet_i\right).$$

Where exp is the usual exponential function which exists since $G(\mathcal{B}_1)$ is nilpotent. This path is not necessarily γ Hölder continuous in $G(\mathcal{B}_1)$ however it is in the quotient group $G(\mathcal{B}_1)/K_1$ where

$$K_1 = \exp\left([\mathcal{B}_1, \mathcal{B}_1]_{\otimes}\right)$$

With

$$[\mathcal{B}_1, \mathcal{B}_1]_{\otimes} = \text{span}\{\bullet_i \otimes \bullet_j - \bullet_j \otimes \bullet_i : i, j = 1 \dots d\}$$

Now we will apply the following extension theorem :

Theorem 5. *Let $\gamma \in (0,1)$ such that $\gamma^{-1} \notin \mathbb{N} \setminus \{0,1\}$. Suppose K is a normal subgroup of $G^{(N)}(V)$. If $\mathbb{X}$ is a γ-Hölder continuous path in the quotient $G^{(N)}(V)/K$, then there exists a γ-Hölder continuous path $\bar{\mathbb{X}}$ taking values in $G^{(N)}(V)$ and satisfying*

$$\pi_{G^{(N)}(V)/K}\left(\bar{\mathbb{X}}\right) = \mathbb{X},$$

where π denotes the projection map.

and obtain a path in $G^{(N)}(\mathcal{B}_1)$. Now let $\mathcal{T}_{(2)}$ be the set of trees with two vertices. Let τ be the tree with two vertices labeled a and b. We set $\bar{X}_{st}^{\tau} = \langle \mathbb{X}_{st}, \tau \rangle - \langle \bar{\mathbb{X}}_{st}^{1}, \bullet_a \otimes \bullet_b \rangle$ Then we can define

$$\mathbb{X}_t = \exp\left(\log \bar{\mathbb{X}}_t^1 + \sum_{\tau \in \mathcal{T}_{(2)}} X_t^{\tau} \tau\right)$$

To make this Hölder continuous we again divide out a subgroup and then apply the extension theorem . We iterate this step for bigger trees until we are done.

References

[1] Hairer, Martin and Kelly, David. "Geometric versus non-geometric rough paths." Ann. Inst. Henri Poincaré, Probab. Stat. Vol. 51 No. 1 (2015): 207-251.

Nikolay Barashkov, University of Bonn
email: **s6nibara@uni-bonn.de**

18 Quasilinear SPDEs via rough paths

after F. Otto and H. Weber [1]
A summary written by Pavel Zorin-Kranich

Abstract

18.1 Introduction

The stated purpose of the article [1] it to treat the *quasilinear* (that is, with the leading order coefficient a depending on u) PDE

$$\partial_2 u - a(u)\partial_1^2 u = P\sigma(u)f, \tag{1}$$

under weak regularity hypotheses on f. Here, the given f and the unknown u are 1-periodic functions on $\mathbb{R}^2$ and $P\zeta = \zeta - \int_{[0,1)^2} \zeta$ is the projection on the space of functions with vanishing average. The functions a, σ are assumed to be C^2 and a is assumed to be uniformly elliptic in the sense $0 < \lambda \leq a \leq 1$. The function f is thought of as coming from a stochastic process, but the stochastic process enters the picture only through a number of assumptions on f that are almost surely satisfied if f is a Gaussian random field with suitable covariance. We will not address the stochastic aspects of this problem in this summary, and for notational convenience the function f will stay fixed throughout. We make the qualitative assumption that f is smooth and for notational convenience we also assume that it has vanishing average.

In order to even formulate the main result we have to consider first the constant coefficient linear PDE

$$(\partial_2 - a_0\partial_1^2)v(\cdot, a_0) = f, \quad \lambda \leq a_0 \leq 1. \tag{2}$$

For smooth f it has a unique solution with vanishing average for every a_0 that can be obtained by applying a suitable Fourier multiplier to f.

The Hölder seminorm $[\cdot]_\alpha$ will be defined with respect to an anisotropic metric:

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{d^\alpha(x,y)}, \quad d(x,y) = |x_1 - y_1| + \sqrt{|x_2 - y_2|}.$$

The supremum norm is denoted by $\|\cdot\|$. Also, we will use the semigroup of mollifications $f_T = \exp(-T(\partial_1^4 - \partial_2^2))f$.

Proposition 1 ([1, Proposition 1])**.** *Let* $\frac{2}{3} < \alpha < 1$ *and suppose*

$$\sup_{T \le 1} (T^{\frac{1}{4}})^{2-\alpha} \|f_T\| \le N_0 \ll 1. \tag{3}$$

Suppose also

$$\|[v(\cdot, a_0'), (\cdot)_T]F\|, \|[\frac{\partial v}{\partial a_0}(\cdot, a_0'), (\cdot)_T]F\| \le N_0^2 (T^{\frac{1}{4}})^{2\alpha-2}$$
$$\textit{for all} \quad T \le 1,\ a_0' \in [\lambda, 1] \tag{4}$$

for all $F = f, \partial_1^2 v(\cdot, a_0), \partial_1^2 \frac{\partial v}{\partial a_0}(\cdot, a_0)$ *with* $a_0 \in [\lambda, 1]$.

Then there exists a 1-periodic solution u *of* (1) *with*

$$[u]_\alpha \lesssim N_0.$$

The main step of the proof is in a sense Lemma 5, because this is the place where the PDE is used. In line with the usual approach to quasilinear PDEs that Lemma treats the functions $a(u)$ and $\sigma(u)$ as some given measurable functions. However, the hypothesis (3) is only a sort of negative exponent Hölder continuity, so the function f is too irregular for standard theory to apply Instead, one deals with a mollified version of the PDE, which introduces a number of commutators that have to be controlled externally. This is the purpose of Lemma 6 and Lemma 7. The proofs of these results are too long for this summary, and we will only indicate how they can be bootstrapped to yield Proposition 1.

18.2 Tools

18.2.1 Controlled solutions

The outermost bootstrapping step in the proof of Proposition 1 goes over the constant N in the following definition of "modeling".

Definition 2 ([1, Definition 1])**.** *Let* $\frac{1}{2} < \alpha < 1$. *We say that* u *is modeled after* (v, x_1) *according to* (μ, a, ν) *with constant* N *provided we have for all points* $x, y \in \mathbb{R}^2$

$$|(u(y) - u(x)) - \mu(x)(v(y, a(x)) - v(x, a(x))) - \nu(x)(y - x)_1| \le N d^{2\alpha}(y, x).$$

This "modeling" is well-behaved under C^2 deformations:

Lemma 3 ([1, Lemma 1]). *Suppose that u is modeled after (v, x_1) according to (μ, a, ν) with constant N. Let also $\sigma : \mathbb{R} \to \mathbb{R}$ be a function with $\|\sigma\|_{C^2} \leq 1/\lambda$. Then $\sigma(u)$ is modeled after (v, x_1) according to $(\sigma'(u)\mu, a, \sigma'(u)\nu)$ with constant*

$$N_\sigma \lesssim N + [u]_\alpha^2.$$

Moreover, it controls the Hölder norm.

Lemma 4 ([1, Corollary 1]). *In the setting of Definition 2, assuming* (3), *we have*

$$[u]_\alpha \lesssim N + N_0\|\mu\|.$$

18.2.2 PDE ingredient

Lemma 5 ([1, Lemma 4]). *Let $0 < \lambda \leq 1$ and $\frac{1}{2} < \alpha < 1$. Suppose that the 1-periodic functions $a \in [\lambda, 1]$, σ, f, and u are related by*

$$\partial_2 u - a\partial_1^2 u = P\sigma f.$$

Suppose furthermore (3) *and that we control the commutators*

$$\|[a, (\cdot)_T]\partial_1^2 u\|, \|[\sigma, (\cdot)_T]f\| \leq N(T^{\frac{1}{4}})^{2\alpha-2} \quad \text{for } T \leq 1 \tag{5}$$

for some constant N. Then u is modeled after (v, x_1) according to (σ, a, ν) for some 1-periodic ν with constant

$$\lesssim_{\alpha,\lambda} N + [a]_\alpha [u]_\alpha + N_0([\sigma]_\alpha + \|\sigma\|[a]_\alpha).$$

18.2.3 Commutator estimates

Lemma 6 ([1, Corollary 2]). *Suppose $\alpha > \frac{2}{3}$. Let u be modeled after (v, x_1) according to (μ, a, ν) with constant N. Suppose that for some function b, we control the commutators*

$$\|[b, (\cdot)_T]\partial_1^2 v(\cdot, a_0)\| \leq N_0^2 (T^{\frac{1}{4}})^{2\alpha-2}, \tag{6}$$

$$\|[b, (\cdot)_T]\partial_1^2 \frac{\partial v}{\partial a_0}(\cdot, a_0)\| \leq N_0^2 (T^{\frac{1}{4}})^{2\alpha-2} \tag{7}$$

for all $0 < T \leq 1$ and $a_0 \in [\lambda, 1]$ for some constant N_0. Then

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[b, (\cdot)_T]\partial_1^2 u\| \lesssim N[b]_\alpha + N_0^2(\|\mu\|[a]_\alpha + [\mu]_\alpha + \|\mu\|).$$

Lemma 7 ([1, Corollary 3])**.** *Suppose $\alpha > \frac{2}{3}$. Let b be modeled after (v, x_1) according to (μ, a, ν) with constant N. Suppose*

$$\sup_{T \le 1} (T^{\frac{1}{4}})^{2-\alpha} \|F_T\| \le N_0 \tag{8}$$

and (4). *Then*

$$\sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[b, (\cdot)_T]F\| \lesssim N_0\big(N + N_0([\mu]_\alpha + \|\mu\|[a]_\alpha + \|\mu\|)\big).$$

18.2.4 Estimates for the constant coefficient PDE

It follows from [1, Lemma 8] and some auxiliary arguments that (3) continues to hold (up to an absolute constant) with f replaced by $\partial_1^2 v(\cdot, a_0)$ or $\partial_1^2 \frac{\partial v}{\partial a_0}(\cdot, a_0)$ with any $a_0 \in [\lambda, 1]$. Moreover,

$$[v(\cdot, a_0)]_\alpha, [\frac{\partial v}{\partial a_0}(\cdot, a_0)]_\alpha \lesssim N_0, \quad \lambda \le a_0 \le 1.$$

18.3 Main bootstrapping loop

It is claimed that for smooth f the equation (1) has a solution u that depends continuously on f. Let N be (nearly) the best constant for which there exists ν such that this solution is modeled after (v, x_1) according to (σ, a, ν) with the constant N, where $a = a(u)$ and $\sigma = \sigma(u)$ are 1-periodic functions of two variables and should not be confused with the functions of one variable a, σ in (1). In order to prove Proposition 1 we will prove an a priori estimate on N in terms of N_0.

By Lemma 4 the α-Hölder seminorm of u, and hence also of any of the related quantities $a, \sigma, a'(u)\sigma, \sigma'(u)a$ is $\lesssim N + N_0$.

By Lemma 3 the functions a, σ are also modeled after (v, x_1) according to $(a'(u)\sigma, a, a'(u)\nu)$ and $(\sigma'(u)\sigma, a, \sigma'(u)\nu)$, respectively, with certain constants N_a, N_σ.

First we apply Lemma 7 with $b = \sigma$, $F = f$. This gives the second hypothesis in (5) with a constant

$$\begin{aligned}\lesssim N_0(N_\sigma + N_0(1 + N_0 + N))& \\ \lesssim N_0(N + (N + N_0)^2 + N_0(1 + N_0 + N))& \\ \lesssim N_0(N + N_0 + N^2),&\end{aligned}$$

where we have used the hypothesis $N_0 \lesssim 1$ in the last step.

Next, we apply Lemma 7 with $b = a$ and both $F = \partial_1^2 v(\cdot, a_0)$ and $F = \partial_1^2 \frac{\partial v}{\partial a_0}(\cdot, a_0)$. This gives the hypotheses (6) and (7), respectively, of Lemma 6 with a constant

$$\lesssim \sqrt{N_0(N + N_0 + N^2)}$$

That lemma in turn yields the first hypothesis in (5) with a constant

$$\lesssim N(N + N_0) + N_0(N + N_0 + N^2)(1 + N + N_0).$$

Applying now Lemma 5 and using again $N_0 \lesssim 1$ we obtain

$$N \lesssim N(N + N_0) + N_0(N + N_0 + N^2)(1 + N) + (N + N_0)^2 + N_0(N + N_0) \\ \lesssim N_0 N^3 + (N + N_0)^2.$$

Since $N_0 \ll 1$ is assumed to be very small, the linear term in N on the right-hand side can be absorbed into the left-hand side, so

$$N \lesssim N_0 N^3 + N^2 + N_0^2,$$

or in other words

$$A \lesssim N_0^3 A^3 + N_0 A^2 + N_0, \quad A = N/N_0.$$

This shows that A must be either $\lesssim N_0$ or $\gtrsim 1/N_0$ (in fact the article [1] states a different bootstrapping inequality [1, (84)], but the two seem to lead to the same results). If the function f vanishes identically, then $u = 0$ solves the equation (1), and in this case $A = 0$. Replacing f by θf, $\theta \in [0, 1]$ it is then claimed that there is a continuous θ-dependent family of solutions for which A also depends continuously on θ, so in fact there is a solution for which the first alternative ($A \lesssim N_0$, hence $N \lesssim N_0^2$, and consequently $[u]_\alpha \lesssim N_0$ by Lemma 4) holds.

References

[1] Otto, F. and Weber, H., *Quasilinear SPDEs via rough paths*. Preprint, arxiv:1605.09744v1

Pavel Zorin-Kranich, University of Bonn
email: **pzorin@math.uni-bonn.de**

19 The continuous Anderson hamiltonian in dimension two

after R. Allez and K. Chouk [1]
A summary written by Irina Holmes

Abstract

We define the Anderson hamiltonian on the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. This operator is formally defined as $\mathcal{H} := -\Delta + \xi$ where Δ is the Laplacian operator and where ξ belongs to a general class of singular potential which includes the Gaussian white noise distribution. We use the notion of paracontrolled distribution as introduced by Gubinelli, Imkeller and Perkowski in [2]. We are able to define the Schrödinger operator $\mathcal{H}$ as an unbounded self-adjoint operator on $L^2(\mathbb{T}^2)$ and we prove that its real spectrum is discrete with no accumulation points for a general class of singular potential ξ. We also establish that the spectrum is a continuous function of a sort of enhancement $\Xi(\xi)$ of the potential ξ. As an application, we prove that a correctly renormalized smooth approximation $\mathcal{H}_\epsilon := -\Delta + \xi_\epsilon + c_\epsilon$ (where ξ_ϵ is a smooth mollification of the Gaussian white noise ξ and c_ϵ an explicit diverging renormalization constant) converge in the sense of the resolvent towards the singular operator $\mathcal{H}$. In the case of a Gaussian white noise ξ, we obtain exponential tail bounds for the minimal eigenvalue (sometimes called ground state) of the operator $\mathcal{H}$ as well as its order of magnitude $logL$ when the operator is considered on a large box $\mathbb{T}_L := \mathbb{R}^2/(L\mathbb{Z}^2)$ with $L \to \infty$.

19.1 Introduction: The parabolic Anderson model (PAM)

The PAM on $\mathbb{R}^d$ can be formulated as

$$\frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) = u(t,x)V(x), \tag{1}$$

for $(t,x) \in (0,\infty) \times \mathbb{R}^d$, subject to

$$u(0,x) = u_0(x), \text{ for } x \in \mathbb{R}^d,$$

where V is a random field on $\mathbb{R}^d$. One of the most important questions about solutions to the PAM is describing their long-term behavior. This is determined by the spectral properties of the Anderson hamiltonian, the operator defined as $\mathcal{H} := -\Delta + V$.

Most instances of the random potential V studied in the literature are highly regular, with a large and even infinite correlation length. The case of entirely uncorrelated potentials poses many challenges, and its study, while still in its early stages, is now a highly active research area. The paper [1] summarized here is a recent development in this area.

The most natural such rough potential is $V = \xi$, where $\xi(x)$, $x \in \mathbb{R}^d$ is a centered Gaussian process with covariance

$$\mathbb{E}[\xi(x)\xi(y)] = \delta(x - y).$$

Solving the PAM in this setting is very difficult for dimensions $d \geq 2$, because of the high degree of irregularity of the random potential – for one, this requires renormalization procedures. So the standard techniques of stochastic PDE theory do not really apply to this outlier case. However, this problem has recently found a framework that can work with it, in the development of rough paths and paracontrolled distributions. The satisfactory construction (almost surely on the whole state space $\mathbb{R}^d$) of solutions to (1) occurred in [3] for $d = 2$ and in [4] for $d = 3$.

19.2 Main results

The paper [1] continues the study of PAM with white noise potential by answering some spectral theoretic questions about the Anderson hamiltonian

$$\mathcal{H} = -\Delta + \xi,$$

with ξ Gaussian white noise. However, the setting here is only the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ in $\mathbb{R}^2$ rather than the whole space. Specifically, consider

$$\partial_t u(t,x) - \Delta u(t,x) = u(t,x)\xi(x), \;\; u(0,x) = u_0(x), \tag{2}$$

where $x \in \mathbb{T}^2$, the function $u_0 \in L^2(\mathbb{T}^2)$, and Δ denotes the Laplacian operator with periodic boundary conditions and ξ is a real white noise distribution on $\mathbb{T}^2$. The solution to (2) may be expressed in terms of the operator $\mathcal{H}$ as

$$u(t,x) = \exp(-t\mathcal{H})u_0(x) := \sum_{n=0}^{\infty} \exp(-t\Lambda_n) \left\langle e_n, u_0 \right\rangle_{L^2(\mathbb{T}^2)} e_n(x), \tag{3}$$

for $x \in \mathbb{T}^2$, with two big assumptions: first, that one can define the operator $\mathcal{H}$ in the first place, and, second, prove that its spectrum Λ_n is real and discrete, with associated orthonormal eigenvectors $e_n \in L^2(\mathbb{T}^2)$. These facts are far from trivial to establish in dimension $d = 2$, where the high irregularity of ξ makes $\mathcal{H}$ ill-defined, and they are the main subject of [1].

The main aspects of the results in [1] are:

1. Construction of the Anderson hamiltonian $\mathcal{H}$ in the continuous setting, with Gaussian white noise (in a finite volume though, restricted to the 2-dimensional torus);

2. Spectral theoretic aspects of $\mathcal{H}$: establish that its spectrum is real and discrete, with an orthonormal family of eigenvectors in $L^2(\mathbb{T}^2)$;

3. Partial results on the limiting statistics of its ground state (the minimal eigenvalue) of $\mathcal{H}$ in a large torus of side length $L \to \infty$.

19.2.1 Analytical results

The first stage of constructing $\mathcal{H}$ is purely analytic. The following result constructs the Schrödinger operator $\mathcal{H}$ for a general class of rough potentials ξ living in a space of Hölder distributions. Here ξ is a general rough distribution living in a *Sobolev space* with index $\alpha < -1$ defined as

$$H^\alpha(\mathbb{T}_L^2, \mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{T}_L, \mathbb{R}) : \sum_{k \in \mathbb{Z}_L^2} (1 + |k|^2)^\alpha |\widehat{f}(k)|^2 < +\infty\},$$

where $\mathbb{T}_L^2 := \mathbb{R}^2/(L^{-1}\mathbb{Z}^2)$ is the torus of size L, $\mathcal{S}'$ is the Schwartz space of tempered distributions, and $\widehat{f}(k)$ is the kth Fourier coefficient. Since the interest later shifts to the limiting spectral properties of $\mathcal{H}$ when considered on a large volume, the result is stated on $\mathbb{T}_L^2$:

Theorem 1. *Let $\alpha \in \left(-\frac{4}{3}, -1\right)$. Then there exists a Banach space $\mathcal{X}^\alpha(\mathbb{T}_L^2) \subset \mathcal{C}^\alpha(\mathbb{T}_L^2) \times \mathcal{C}^{2\alpha+2}(\mathbb{T}_L^2)$ such that for all $\Xi = (\epsilon, \Xi_2) \in \mathcal{X}^\alpha$, there exists a Hilbert space $\mathcal{D}_\Xi \subset L^2(\mathbb{T}_L^2)$ (which is dense in $L^2(\mathbb{T}_L^2)$) and a unique self-adjoint operator $\mathcal{H}(\Xi) : \mathcal{D}_\Xi \to L^2(\mathbb{T}_L^2)$ with the following properties:*

1. *If ξ is a smooth function, then we can choose Ξ_2 such that:*

$$\mathcal{D}_{(\xi, \Xi_2 + c)} = H^2(\mathbb{T}_L^2), \quad \mathcal{H}(\Xi) f = -\Delta f + f(\xi + c)$$

for all $f \in H^2(\mathbb{T}_L^2)$ and $c \in \mathbb{R}$.

2. *The spectrum* $(\Lambda_n(\Xi))_{n\in\mathbb{N}^*}$ *of* $\mathcal{H}(\Xi)$ *is real, discrete without any accumulation point and satisfy* $\Lambda_n(\Xi) \to +\infty$ *when* $n \to \infty$,

$$\Lambda_1(\Xi) \leq \Lambda_2(\Xi) \leq \ldots \leq \Lambda_n(\Xi)$$

and $dim(\Lambda_n(\Xi) - \mathcal{H}(\Xi)) < +\infty$. *Moreover,* $L^2(\mathbb{T}_L^2) = \oplus_n ker(\Lambda_n(\Xi) - \mathcal{H}(\Xi))$.

3. *The eigenvalues* $(\Lambda_n)_{n\in\mathbb{N}}$ *are the solution of a min-max principle.*

4. *For each* $n \in \mathbb{N}$, *the map* $\Xi \to \Lambda_n(\Xi)$ *is locally-Lipschitz. More precisely, there exist two positive constants* C *and* M *which do not depend on* L *such that, for all* $\alpha \in (-4/3, -1)$, $\gamma < \alpha + 2$, $n \in \mathbb{N}$, $\Xi, \widetilde{\Xi} \in \mathcal{X}^\alpha$,

$$|\Lambda_n(\Xi) - \Lambda_n(\widetilde{\Xi})| \leq$$
$$\leq Cn\left(1 + n^{\frac{2\gamma-\alpha}{\alpha+2}} + (1 + \Lambda_n(0))^{2\gamma}\right)^2 \|\Xi - \widetilde{\Xi}\|_{\mathcal{X}^\alpha}(1 + \|\widetilde{\Xi}\|_{\mathcal{X}^\alpha} + \|\Xi\|_{\mathcal{X}^\alpha})^M,$$

where $\Lambda_n(0)$ *is the* n*-lowest eigenvalue of the Laplacian operator* $-\Delta$.

5. *For all* $a \in \mathbb{R} \setminus \{\Lambda_n(\Xi), n \geq 1\}$, *the resolvent map* $\Xi \to \mathcal{G}_a(\Xi) = (a + \mathcal{H}(\Xi))^{-1}$ *is locally Lipschitz.*

Remarks about this theorem:

- The important point is that knowledge of ξ alone is not enough to define $\mathcal{H}$ as an unbounded operator on $L^2(\mathbb{T}^2)$ - this is where the quantity Ξ_2 comes in. Roughly speaking, this is the ill-defined part of the product $\xi(1-\Delta)^{-1}\xi$, living in $\mathcal{C}^{2\alpha+2}(\mathbb{T}^2)$, where $\mathcal{C}^\alpha$ denotes *the Hölder-Besov space.*

- Then $\Xi := (\xi, \Xi_2)$, an enhancement of the rough distribution ξ containing the additional information Ξ_2, can be used to make sense of the ill-defined product ξf of two distributions, and define $\mathcal{H}$ on an explicit domain $\mathcal{D}_\Xi \subset L^2(\mathbb{T}^2, \mathbb{R})$.

19.2.2 Stochastic results

The focus is now to show that the Gaussian white noise fits in the analytical framework developed above. The authors use classical stochastic analysis techniques to prove that one can construct Ξ_2 in this setting in a robust way through smooth approximations. Note that the Gaussian white noise ξ fits the description in Theorem 1 since $\xi \in \mathcal{C}^\alpha$ a.s. for all $\alpha < -1$. In this case, Theorem 1 shows that $\mathcal{H}$ as a discrete real spectrum continuous with respect to the *enhanced Gaussian noise* (ξ, Ξ_2). The next result is a smooth approximation result for the operator $\mathcal{H}$ associated with ξ. Below, $\widehat{\theta}_\epsilon = \epsilon^{-2}\widehat{\theta}(\cdot/\epsilon)$ is an approximation of the identity:

Theorem 2. *Let $\alpha < -1$, ξ be a Gaussian white noise, $\xi_\epsilon := \xi \star \widehat{\theta}_\epsilon$ be a smooth mollification of $\xi \in \mathcal{C}^\alpha$ and Ξ_2^ϵ as given in Theorem 1 such that $\mathcal{H}(\xi_\epsilon, \Xi_2^\epsilon) = \mathcal{H}_\epsilon$, where*

$$\mathcal{H}_\epsilon := -\Delta + \xi_\epsilon.$$

Then, there exists $\Xi^{wn} = (\xi, \Xi_2^{wn}) \in \mathcal{X}^\alpha(\mathbb{T}_L^2)$ and a constant $c_\epsilon := c_\epsilon(\theta) \to +\infty$ as $\epsilon \to 0$ such that the following convergence holds

$$(\xi_\epsilon, \Xi_2^\epsilon + c_\epsilon) \to_{\epsilon \to 0} (\xi, \Xi_2^{wn})$$

in $L^p(\Omega, \mathcal{C}^\alpha \times \mathcal{C}^{2\alpha+2})$ for all $p > 0$ and almost surely in $\mathcal{C}^\alpha \times \mathcal{C}^{2\alpha+2}$. Moreover, the limiting distribution Ξ does not depend on the mollification function θ and the normalizing constant c_ϵ has the following asymptotic expansion

$$c_\epsilon = \frac{1}{2\pi} log\left(\frac{1}{\epsilon}\right) + O(1),$$

where $O(1)$ refers to any fixed constant, independent of ϵ.

Theorems 1 and 2 now allow one to define the Schrödinger operator $\mathcal{H}$ associated with the Gaussian noise potential ξ by setting

$$\mathcal{H} := \mathcal{H}(\Xi^{wn}).$$

To make sense of this definition, the next result establishes convergence in the sense of the resolvent (convergence of the spectrum) of the smooth approximations $\mathcal{H}_\epsilon + c_\epsilon$ as defined in Theorem 2 towards the operator $\mathcal{H}$:

Theorem 3. *With the same notations as in Theorem 2, we denote by*

$$\Lambda_1^\epsilon \leq \Lambda_2^\epsilon \leq \dots$$

the eigenvalues of the operator $\mathcal{H}_\epsilon$. Then, for any $n \in \mathbb{N}$, almost surely,

$$\Lambda_n^\epsilon + c_\epsilon \to_{\epsilon \to 0} \Lambda_n(\Xi^{wn}),$$

where $(\Lambda_n(\Xi^{wn}))_{n \in \mathbb{N}}$ denotes the discrete set of the eigenvalues of $\mathcal{H}(\Xi^{wn})$.

Finally, the authors obtain partial results about the limiting spectral statistics of the operator $\mathcal{H}(\Xi^{wn})$ when the volume of the torus tends to ∞. Specifically, the next result provides an upper bound on the asymptotic order of the ground state in the limit of large volume $L \to \infty$.

Theorem 4. *For any $n \in \mathbb{N}$ and $p \geq 1$,*

$$\sup_{L>0} \mathbb{E}\left[\left|\frac{\Lambda_n(\Xi^{wn})}{logL}\right|^p\right] < +\infty.$$

Besides, there exist two positive constants C_1 and C_2 such that for any $x < 0$, we have

$$e^{C_2 x} \leq \mathbb{P}(\Lambda_1(\Xi^{wn}) \leq x) \leq e^{C_1 x}.$$

References

[1] Allez, R. and Chouk, K., *The continuous Anderson hamiltonian in dimension two.* Preprint arXiv:1511.02718 [math.PR];

[2] Gubinelli, M., Imkeller, P. and Perkowski, *Paracontrolled distributions and singular PDEs.* arXiv:1210.2684, Forum of Mathematics, Pi (2015);

[3] Hairer, M. and Labbé, C., *A simple construction of the continuum parabolic Anderson model on $\mathbb{R}^2$.* Elec. Comm. Probab. 20, 1–11 (2015);

[4] Hairer, M. and Labbé, C., *Multiplicative stochastic heat equations on the whole space.* Preprint arXiv:1504.07162 [math.AP];

[5] König, W., *The Parabolic Anderson Model: Random Walk in Random Potential.* Birkhäuser (2016).

Irina Holmes, Washington University in St. Louis
email: **irina.holmes@email.wustl.edu**