HOCHSCHILD COHOMOLOGY OF GENTLE ALGEBRAS

SEFI LADKANI

ABSTRACT. We compute the Hochschild cohomology groups of gentle algebras and show that they are determined by the derived invariant introduced by Avella-Alaminos and Geiss.

Gentle algebras are certain finite-dimensional algebras defined combinatorially in terms of quivers with relations. They have remarkable homological properties; they are Gorenstein [9] and the class of gentle algebras is closed under derived equivalence [13]. It is a long standing problem to classify gentle algebras up to derived equivalence. To this end, one searches for derived invariants that will allow to distinguish the different derived equivalence classes.

One such invariant was developed by Avella-Alaminos and Geiss [3]. It takes the form of a function $\phi_\Lambda : \mathbb{N}^2 \to \mathbb{N}$ that can be effectively computed from the quiver with relations of a gentle algebra Λ. In some cases this invariant is fully capable of distinguishing the derived equivalence classes [2, 3, 7], but in general it is not complete and thus further derived invariants are needed.

A possible candidate is the Hochschild cohomology, which is well-known to be derived invariant [10, 12], defined as $\mathrm{HH}^*(\Lambda) = \mathrm{Ext}^*_{\Lambda^{op} \otimes_K \Lambda}(\Lambda, \Lambda)$ for an algebra Λ over a field K. Indeed, in [6] it was used, together with other derived invariants, to classify up to derived equivalence the gentle algebras with two vertices and those with three vertices and zero Cartan determinant.

However, in this note we show that at least as a graded vector space, the Hochschild cohomology of a gentle algebra is completely determined by its Avella-Alaminos-Geiss derived invariant and thus cannot distinguish derived classes not already distinguished by the latter. In fact, we give an explicit formula for the dimensions of the Hochschild cohomology groups in terms of the Avella-Alaminos-Geiss derived invariant, based on the projective resolution of a monomial algebra as a bimodule over itself given by Bardzell [4] (see also [14]), observing that gentle algebras are quadratic monomial algebras.

In order to formulate our results more precisely, we encode the dimensions of the Hochschild cohomology groups of a finite-dimensional algebra Λ over a field K in a formal power series

$$h_\Lambda(z) = \sum_{i=0}^{\infty} \dim_K \mathrm{HH}^i(\Lambda) \cdot z^i - 1$$

and we define, for $n \geq 1$, the formal power series

$$g_n(z) = \frac{z^n(1+z)(\varepsilon_n + z^n)}{1 - z^{2n}}, \quad \text{where} \quad \varepsilon_n = \begin{cases} 1 & \text{if } n \text{ is even or char } K = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that for a finite quiver Q, the *Euler characteristic* $\chi(Q)$ is defined as the number of its vertices minus the number of its arrows.

Notations. For a gentle algebra Λ, denote by Q_Λ its quiver and by ϕ_Λ its Avella-Alaminos-Geiss derived invariant. We assume that all algebras are *connected*.

Theorem 1. *Let Λ be a gentle algebra. Then*
$$h_\Lambda(z) = \big(1 - \chi(Q_\Lambda)\big)z + \sum_{m \geq 0} \phi_\Lambda(1, m)z^m + \sum_{m > 0} \phi_\Lambda(0, m)g_m(z)$$

Remark. The Euler characteristic $\chi(Q_\Lambda)$ can be expressed in terms of ϕ_Λ by the formula
$$2 \cdot \chi(Q_\Lambda) = \sum_{(n,m) \in \mathbb{N}^2} \phi_\Lambda(n, m)(n - m),$$
hence $h_\Lambda(z)$ depends only on ϕ_Λ and whether or not the field K has characteristic 2.

Let us write explicitly the dimensions of the Hochschild cohomology groups. We define $\psi_\Lambda(n) = \sum_{d|n} \phi_\Lambda(0, d)$ for $n \geq 1$.

Corollary 1. *Let Λ be a gentle algebra. Then*
(a) $\dim \mathrm{HH}^0(\Lambda) = 1 + \phi_\Lambda(1, 0)$.
(b) $\dim \mathrm{HH}^1(\Lambda) = 1 - \chi(Q_\Lambda) + \phi_\Lambda(1, 1) + \begin{cases} \phi_\Lambda(0, 1) & \text{if char } K = 2, \\ 0 & \text{otherwise.} \end{cases}$
(c) $\dim \mathrm{HH}^n(\Lambda) = \phi_\Lambda(1, n) + a_n\psi_\Lambda(n) + b_n\psi_\Lambda(n - 1)$ *for $n \geq 2$, where*
$$(a_n, b_n) = \begin{cases} (1, 0) & \text{if char } K \neq 2 \text{ and } n \text{ is even,} \\ (0, 1) & \text{if char } K \neq 2 \text{ and } n \text{ is odd,} \\ (1, 1) & \text{if char } K = 2. \end{cases}$$

Remark. The numbers $\dim \mathrm{HH}^n(\Lambda)$ can be effectively computed from the quiver with relations of Λ, since the same is true for ϕ_Λ.

Corollary 2. *The following conditions are equivalent for a gentle algebra Λ.*
(i) $\mathrm{HH}^1(\Lambda) = 0$;
(ii) $\mathrm{HH}^i(\Lambda) = 0$ *for all $i \geq 1$;*
(iii) Λ *is piecewise hereditary of Dynkin type A.*

Corollary 3. *Let Λ be a gentle algebra of finite global dimension. Then*
$$h_\Lambda(z) = \big(1 - \chi(Q_\Lambda)\big)z + \sum_{n \geq 0} \phi_\Lambda(1, n)z^n.$$

In particular, $\dim \mathrm{HH}^n(\Lambda) = \phi_\Lambda(1, n)$ for $n > 1$.

As an application of our results, we compute the Hochschild cohomology groups of the gentle algebras arising from surface triangulations introduced by Assem, Brüstle, Charbonneau and Plamondon [1].

A marked bordered oriented surface without punctures is a pair (S, M) where S is a compact, connected Riemann surface with non-empty boundary ∂S and $M \subset \partial S$ is a finite set of *marked points* containing at least one point from each connected component

of ∂S. In [1] the authors associate to each ideal triangulation T of (S, M) a gentle algebra Λ_T. The invariant ϕ_{Λ_T} was computed in [8].

Notations. Denote by g the genus of S, by $b \geq 1$ the number of connected components of its boundary ∂S and by c_1 the number of those components with exactly one marked point. For an ideal triangulation T of (S, M), we set the following quantities:

c_0 – the number of boundary components with exactly two marked points such that their two boundary segments are sides of the same triangle in T;

d – the number of triangles of T such that two of their sides are boundary segments.

In addition we set $f_3(z) = z + g_3(z)$ (this agrees with our notation in [11]).

Theorem 2. *Let T be an ideal triangulation of a marked bordered oriented surface without punctures. Then*

$$h_{\Lambda_T}(z) = c_0 + (2g + b - 1 + c_1)z + \big(4(g - 1) + 2b + d\big)f_3(z).$$

Corollary 4. $\mathrm{HH}^2(\Lambda) = 0$ *for any gentle algebra Λ arising from surface triangulation.*

The above theorem applies in particular to cluster-tilted algebras of affine type $\widetilde{A}$. In [5], the quivers of these algebras have been explicitly described and their derived equivalence classes were characterized in terms of four parameters.

Corollary 5. *Let Λ be a cluster-tilted algebra of type $\widetilde{A}$ with parameters (s_1, t_1, s_2, t_2). Then*

$$h_\Lambda(z) = c_0 + (1 + c_1)z + (t_1 + t_2)f_3(z)$$

where $c_0 = |\{i \,:\, (s_i, t_i) = (0, 1)\}|$ and $c_1 = |\{i \,:\, (s_i, t_i) = (1, 0)\}|$.

References

1. Ibrahim Assem, Thomas Brüstle, Gabrielle Charbonneau-Jodoin, and Pierre-Guy Plamondon, *Gentle algebras arising from surface triangulations*, Algebra Number Theory **4** (2010), no. 2, 201–229.
2. Diana Avella-Alaminos, *Derived classification of gentle algebras with two cycles*, Bol. Soc. Mat. Mexicana (3) **14** (2008), no. 2, 177–216.
3. Diana Avella-Alaminos and Christof Geiss, *Combinatorial derived invariants for gentle algebras*, J. Pure Appl. Algebra **212** (2008), no. 1, 228–243.
4. Michael J. Bardzell, *The alternating syzygy behavior of monomial algebras*, J. Algebra **188** (1997), no. 1, 69–89.
5. Janine Bastian, *Mutation classes of $\tilde{A}_n$-quivers and derived equivalence classification of cluster tilted algebras of type $\tilde{A}_n$*, `arXiv:0901.1515`.
6. Christine Bessenrodt and Thorsten Holm, *q-Cartan matrices and combinatorial invariants of derived categories for skewed-gentle algebras*, Pacific J. Math. **229** (2007), no. 1, 25–47.
7. Grzegorz Bobiński and Piotr Malicki, *On derived equivalence classification of gentle two-cycle algebras*, Colloq. Math. **112** (2008), no. 1, 33–72.
8. Lucas David-Roesler and Ralf Schiffler, *Algebras from surfaces without punctures*, `arXiv:1103.4357`.
9. Ch. Geiß and I. Reiten, *Gentle algebras are Gorenstein*, Representations of algebras and related topics, Fields Inst. Commun., vol. 45, Amer. Math. Soc., Providence, RI, 2005, pp. 129–133.
10. Dieter Happel, *Hochschild cohomology of finite-dimensional algebras*, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Math., vol. 1404, Springer, Berlin, 1989, pp. 108–126.
11. Sefi Ladkani, *Hochschild cohomology of cluster-tilted algebras of finite representation type.*
12. Jeremy Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991), no. 1, 37–48.

13. Jan Schröer and Alexander Zimmermann, *Stable endomorphism algebras of modules over special biserial algebras*, Math. Z. **244** (2003), no. 3, 515–530.
14. Emil Sköldberg, *The Hochschild homology of truncated and quadratic monomial algebras*, J. London Math. Soc. (2) **59** (1999), no. 1, 76–86.

Mathematical Institute of the University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany
  URL: http://www.math.uni-bonn.de/people/sefil
  E-mail address: sefil@math.uni-bonn.de