Universal derived equivalences of posets and applications to cluster-tilting objects

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Posets, diagrams and sheaves

X – *poset* (finite partially ordered set)

$\mathcal{A}$ – abelian category

$\mathcal{A}^X$ – the category of *diagrams* over X with values in $\mathcal{A}$, or *functors* $F : X \to \mathcal{A}$, consisting of:

- An object F_x of $\mathcal{A}$ for each $x \in X$.
- A morphism $r_{xx'} \in \mathrm{Hom}_{\mathcal{A}}(F_x, F_{x'})$ for each $x < x'$.

such that $r_{xx''} = r_{x'x''} r_{xx'}$ for all $x < x' < x''$ (commutativity).

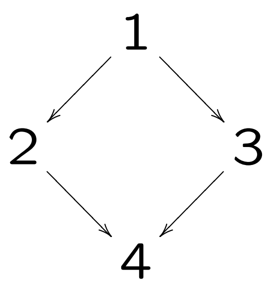
Natural *topology* on X: $U \subseteq X$ is open if $x \in U$, $x \leq x' \Rightarrow x' \in U$

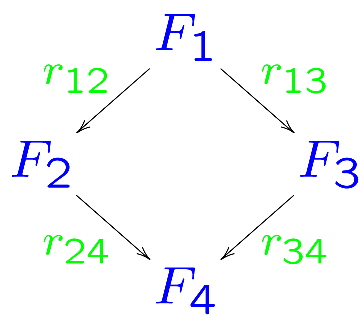
Diagrams can be identified with *sheaves* over X with values in $\mathcal{A}$.

Posets, diagrams and sheaves – Example

Let $X = \{1, 2, 3, 4\}$ with $1 < 2$, $1 < 3$, $1 < 4$, $2 < 4$, $3 < 4$.

A *diagram* over X is shown on the right:





$$r_{24}r_{12} = r_{14} = r_{34}r_{13}$$

The *open sets* are

$$\phi, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}.$$

Derived categories

$\mathcal{B}$ – abelian category, $\mathcal{C}^b(\mathcal{B})$ – the category of *bounded complexes*

$$K^\bullet = \dots \xrightarrow{d} K^{-1} \xrightarrow{d} K^0 \xrightarrow{d} K^1 \xrightarrow{d} \dots$$

with $K^i \in \mathcal{B}$, $d^2 = 0$ and $K^i = 0$ for $|i| \gg 0$.

A morphism $f : K^\bullet \to L^\bullet$ is a *quasi-isomorphism* if

$$H^i f : H^i K^\bullet \to H^i L^\bullet$$

are isomorphisms for all $i \in \mathbb{Z}$.

The *bounded derived category* $\mathcal{D}^b(\mathcal{B})$ is obtained from $\mathcal{C}^b(\mathcal{B})$ by *localization* with respect to the quasi-isomorphisms (that is, we formally invert all quasi-isomorphisms).

Universal derived equivalence

Two posets X and Y are *universally derived equivalent* ($X \overset{u}{\sim} Y$) if

$$\mathcal{D}^b(\mathcal{A}^X) \simeq \mathcal{D}^b(\mathcal{A}^Y)$$

for any abelian category $\mathcal{A}$.

Fix a field k, and specialize:
$\operatorname{mod} k$ – the category of finite dimensional vector spaces over k.

$(\operatorname{mod} k)^X$ can be identified with the category of finitely generated *right modules* over the *incidence algebra* of X over k.

X and Y are *derived equivalent* ($X \sim Y$) if

$$\mathcal{D}^b(\operatorname{mod} kX) \simeq \mathcal{D}^b(\operatorname{mod} kY)$$

Comments on derived equivalence

No known *algorithm* that decides if $X \sim Y$ (or $X \overset{u}{\sim} Y$).

However, one can use:

• *Invariants* of the derived category;
If $X \sim Y$ then X and Y must have the same invariants.

Examples of invariants are:

- The *number of points* of X.
- The *Euler bilinear form* on X, closely related to the *Möbius function* of X.

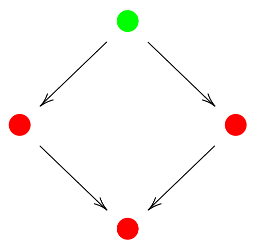
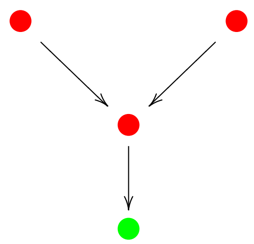
• *Constructions*
Start with some "nice" X and get many Y-s with $X \sim Y$.

Known constructions

• **BGP Reflection**
When X is a tree and $s \in X$ is a *source* (or a *sink*), invert all arrows from (to) s and get a new tree X' with $X' \sim X$.

Example.

• **The square and D_4**

 and  are equivalent.

New construction – Flip-flops

Let $(X, \leq_X)$, $(Y, \leq_Y)$ be posets, $f : X \to Y$ order-preserving.

Define two partial orders $\leq_+^f$, $\leq_-^f$ on $X \sqcup Y$ as follows:

- Keep the original partial orders inside X and Y.
- Add the relations

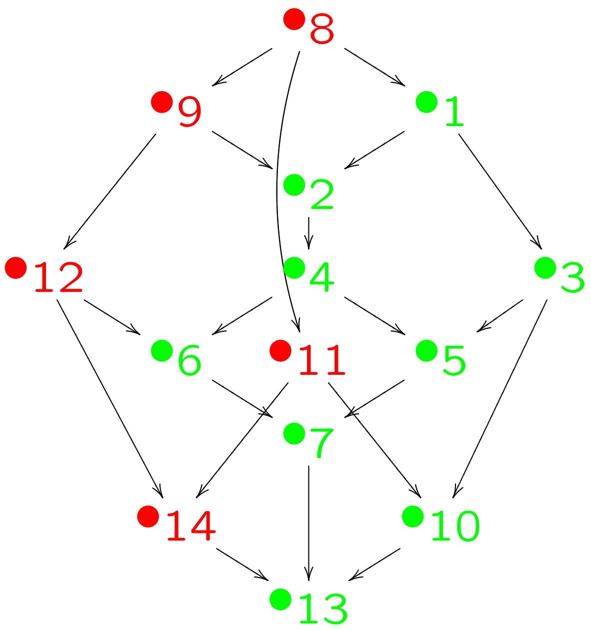
$$x \leq_+^f y \iff f(x) \leq_Y y$$
$$y \leq_-^f x \iff y \leq_Y f(x)$$

for $x \in X$, $y \in Y$.

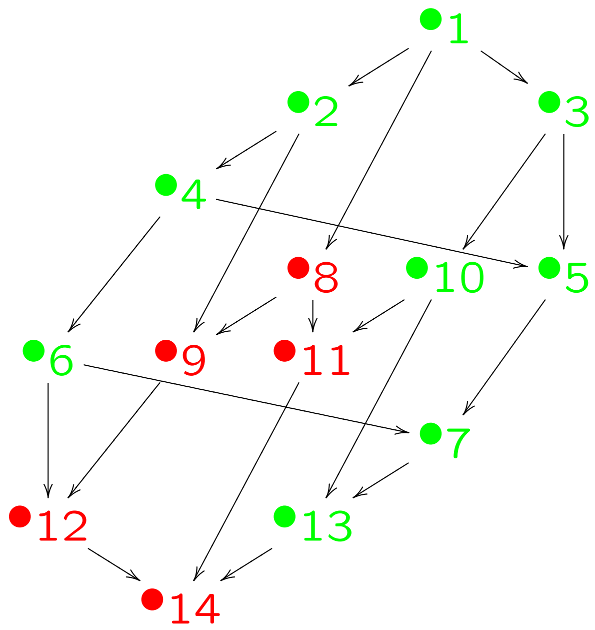
Theorem [L1]. $(X \sqcup Y, \leq_+^f) \overset{u}{\sim} (X \sqcup Y, \leq_-^f)$.

Flip-flop – An example

f : $8 \mapsto 1$ $9 \mapsto 2$ $11 \mapsto 10$ $12 \mapsto 6$ $14 \mapsto 13$



$(X \sqcup Y, \leq_+^f)$



$(X \sqcup Y, \leq_-^f)$

Cluster tilting objects

Q – quiver without oriented cycles with n vertices; k – field

The *cluster category* associated with Q [BMRRT,CCS,FZ] is defined as the orbit category

$$\mathcal{C}_Q = \mathcal{D}^b(\operatorname{mod} kQ)/\nu \cdot [-2]$$

where $\nu : \mathcal{D}^b(\operatorname{mod} kQ) \to \mathcal{D}^b(\operatorname{mod} kQ)$ is the *Serre functor*.

Indecomposables: $\operatorname{ind} \mathcal{C}_Q = \operatorname{ind} kQ \cup \{P_x[1] \,:\, x \text{ is a vertex of } Q\}$.

A (basic) object T of $\mathcal{C}_Q$ is *cluster tilting* if $\operatorname{Hom}_{\mathcal{C}_Q}(T, T[1]) = 0$ and T has n indecomposable summands.

$\mathcal{T}_Q$ – the set of cluster tilting objects in $\mathcal{C}_Q$.

Partial order on the set of cluster tilting objects

For $T = \bigoplus_{i=1}^{n} T_i \in \mathcal{T}_Q$, let $\widehat{T} \in \operatorname{mod} kQ$ be the sum of the T_i which are in $\operatorname{ind} kQ$. Consider the *torsion class*

$$\operatorname{fac} \widehat{T} = \left\{ M \in \operatorname{mod} kQ \; : \; M \text{ is a quotient of } \widehat{T}^m \text{ for some } m \geq 1 \right\}$$

and define [IT] a *partial order* on $\mathcal{T}_Q$ by setting $T \leq T'$ if $\operatorname{fac} \widehat{T} \supseteq \operatorname{fac} \widehat{T'}$.

When Q is a Dynkin diagram of type A, D, or E, the poset $\mathcal{T}_Q$ is known also as a *Cambrian lattice* [R], which is a quotient of the *weak order* on the corresponding Coxeter group.

In type A with the linear orientation, we get the *Tamari lattices*. Their Hasse diagrams are the 1-skeletons of polytopes known as the *Stasheff Associhedra*.

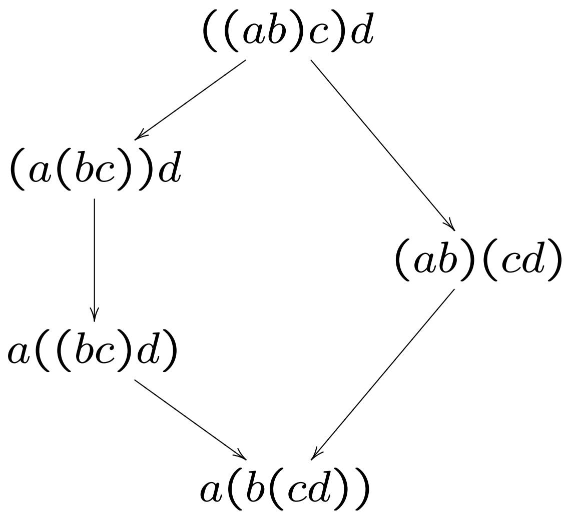
Tamari Lattices for A_1 and A_2

A_1:

$\mathcal{T}_{\bullet}$ $\qquad$ $(ab)c \longrightarrow a(bc)$

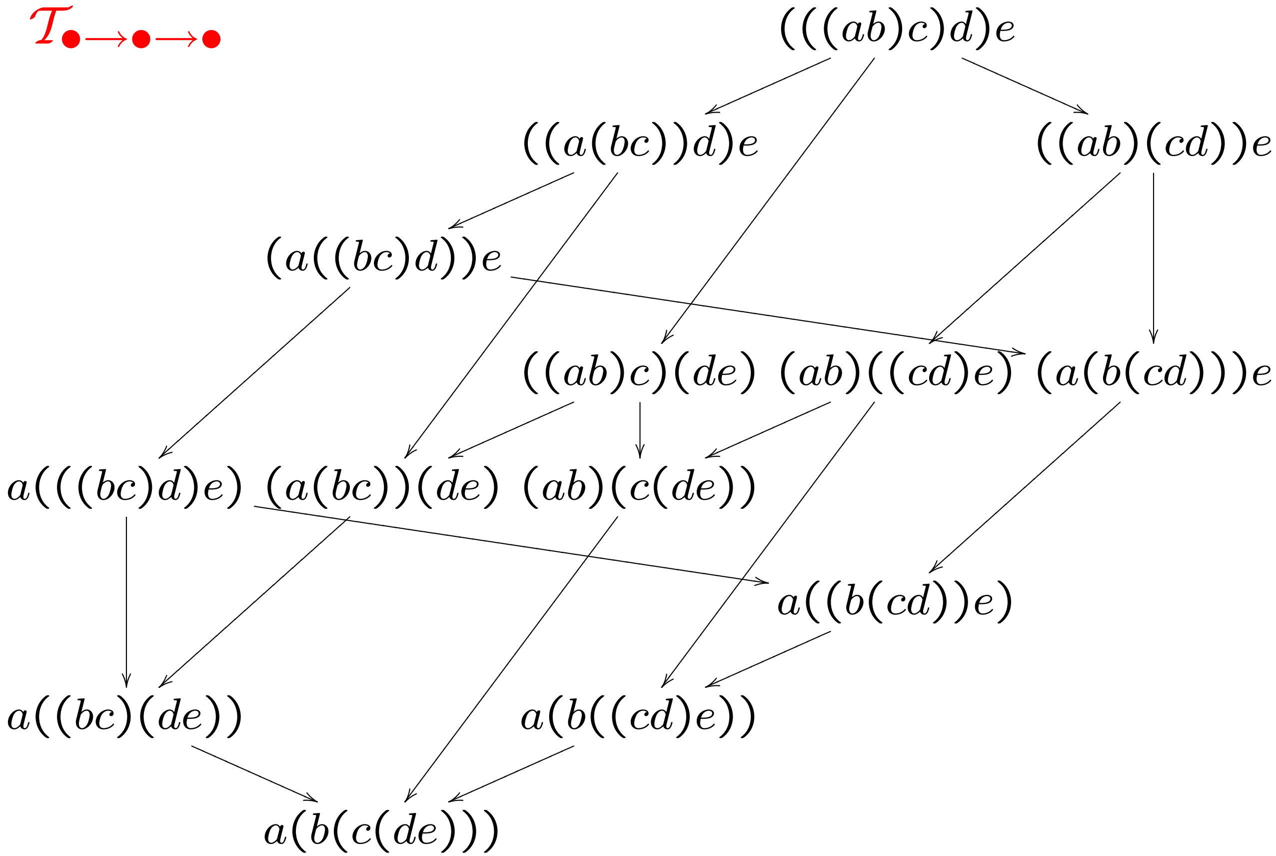
A_2:

$\mathcal{T}_{\bullet \to \bullet}$



Tamari Lattice for A_3

$\mathcal{T}_{\bullet\to\bullet\to\bullet}$



Flip-flops on posets of cluster tilting objects

Q – quiver without oriented cycles; x – a sink in Q.

Q' – the *BGP reflection* with respect to x.

Theorem [L2]. $\mathcal{T}_Q$ and $\mathcal{T}_{Q'}$ are related via a flip-flop.

$$\mathcal{T}_Q \simeq (\mathcal{T}_Q^x \sqcup \mathcal{T}_Q \setminus \mathcal{T}_Q^x, \leq_+^f) \qquad \mathcal{T}_{Q'} \simeq (\mathcal{T}_{Q'}^{x[1]} \sqcup \mathcal{T}_{Q'} \setminus \mathcal{T}_{Q'}^{x[1]}, \leq_-^{f'})$$

$\mathcal{T}_Q^x$ – cluster tilting objects in $\mathcal{T}_Q$ containing P_x as summand.

$\mathcal{T}_{Q'}^{x[1]}$ – cluster tilting objects in $\mathcal{T}_{Q'}$ containing $P_x[1]$ as summand.

f and f' are defined via *cluster mutation*.

Corollary. If $\mathcal{D}^b(\mathrm{mod}\, kQ_1) \simeq \mathcal{D}^b(\mathrm{mod}\, kQ_2)$ then $\mathcal{T}_{Q_1} \overset{u}{\sim} \mathcal{T}_{Q_2}$.

References

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L1 Sefi Ladkani, *Universal derived equivalences of posets*, `arXiv:0705.0946v2`.

L2 Sefi Ladkani, *Universal derived equivalences of posets of cluster tilting objects*, `arXiv:0710.2860`.

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