How pouring honey on a doughnut can help with understanding the solar system

Michael B. Rothgang (he/him)

Wendl group
Humboldt-Universität zu Berlin

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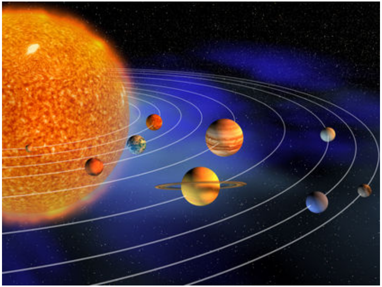
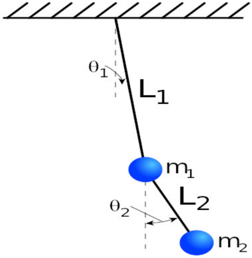
Motivation: two dynamical systems



The solar system (simplified).

Source: http://www.scienceclarified.com/photos/solar-system-2865.jpg



A double pendulum.

Source: By JabberWok, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=1601029

Are such dynamical systems stable?
Do they show chaotic behaviour?
Do they have periodic orbits?

Hamiltonian systems: from Newton's to Hamilton's equations

- system of particles moving with n degrees of freedom

$$q(t) = (q_1(t), \dots q_n(t))$$

- forces are derived from a **potential** $V(q)$ by $F(q) = -\nabla V(q)$
- Newton's second law states $m_i \ddot{q}_j = -\frac{\partial V}{\partial q_j}$

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- Newton's second law states $m_i \ddot{q}_j = -\frac{\partial V}{\partial q_j}$
- Hamilton: consider momenta $p_j := m_j \dot{q}_j$
- total energy defines the Hamiltonian function

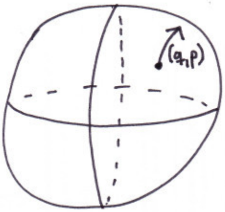
$$H\colon \mathbb{R}^{2n} \to \mathbb{R}, \quad (q,p) \mapsto \underbrace{\sum_{j=1}^{n} \frac{p_j^2}{2m_j}}_{\text{kinetic energy}} + \underbrace{V(q)}_{\text{potential forces}}$$

- Newton's equations become **Hamilton's equations**

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \qquad \text{for } j = 1, \dots n \qquad \text{(H)}$$

Hamilton's equations on a manifold: symplectic manifolds

- key insight: regard $(q(t), p(t))$ as trajectory in **phase space** $\mathbb{R}^{2n} = T^*\mathbb{R}^n$
- double pendulum: rigid arms mean $q(t) = (q_1(t), q_2(t)) \in \mathbb{T}^2$, phase space is cotangent bundle $T^*\mathbb{T}^2$
- for systems with constraints, treat (q, p) as **local coordinates** of a point moving in a manifold



Definition

A $2n$-dimensional manifold is **symplectic** iff it is covered by coordinate charts $(q_1, p_1, \ldots, q_n, p_n)$ such that for all smooth $H\colon M \to \mathbb{R}$, all coordinate changes preserve the form of (H).

Hamilton's equation in symplectic manifolds

Fact

M is symplectic iff M admits a closed non-degenerate 2-form ω.

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Definition

For (M, ω) symplectic, $H\colon M \to \mathbb{R}$ smooth, the **Hamiltonian vector field** X_H of H is defined by $\omega(X_H, \cdot) = -dH$.

Exercise

Solutions (q, p) of (H) are the integral curves of X_H.

Arnold conjecture

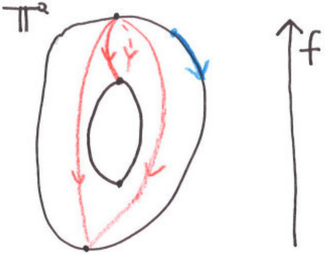
If M is a closed symplectic manifold and $H\colon M \to \mathbb{R}$ smooth, then

$$\#\text{ 1-periodic orbits of } X_H \geq \sum_{i=1}^{n} b_i(M),$$

where $b_i(M) := \operatorname{rk} H_i(M)$ is the i-th Betti number of M.

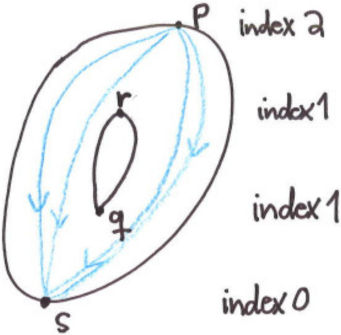
Pouring honey on a donut: gradient flow lines reveal topology!

- Pour honey on doughnut:
  flows along negative gradient
- Four critical points:
  flows stays fixed
- Flow lines tell us about topology:
  e.g. non-contractible loops



Generalising to general smooth manifolds

- M closed smooth manifold, $f: M \to \mathbb{R}$ **Morse** function
- Gradient ∇f for "nice" Riemannian metric on M
- **Morse index** $\mathrm{ind}(p)$ of critical point $p \in M$, $0 \leq \mathrm{ind}(p) \leq \dim M$



For critical points p and q, consider the space

$$\mathcal{M}(p,q) := \Big\{\gamma\colon \mathbb{R} \to M \text{ gradient flow line with } \lim_{t\to-\infty} \gamma(t) = p \text{ and } \lim_{t\to\infty} \gamma(t) = q\Big\}.$$

Key fact

For "almost all" choices of Riemann metric, $\mathcal{M}(p,q)$ is a smooth manifold of dimension $\mathrm{ind}(p) - \mathrm{ind}(q)$.

Morse homology on smooth manifolds

- Is defined via a chain complex $(CM_k(f), \delta)$
- Chain groups $CM_k(f) := \mathbb{Z}_2 \langle \text{critical points of index } k \rangle$
- Differential $\delta_k \colon CM_k(f) \to CM_{k-1}(f)$ defined by

$$\langle p \rangle \mapsto \sum_{\substack{q \text{ critical point} \\ \mathrm{ind}(p) - \mathrm{ind}(q) = 1}} \#_2 \, \mathcal{M}(p,q)/_{\mathbb{R}} \, \langle q \rangle$$

- Check: $(CM_k), \delta)$ is a chain complex, i.e. $\delta^2 = 0$.
  Its homology $HM(M, f)$ is the **Morse homology** of M.

Theorem

Morse homology $HM_*(M, f)$ is isomorphic to the singular homology $H_*(M; \mathbb{Z}_2)$ of M.

Counting periodic Hamiltonian orbits via gradient flows

idea: transfer approach of Morse homology

Principle of least action

1-periodic Hamiltonian orbits are critical points of symplectic action functional $\mathcal{A}_H\colon \{\text{contractible loops in } M\} \to \mathbb{R}$

Define Floer homology $HF(H)$ on closed* symplectic manifolds

- generators: contractible 1-periodic Hamiltonian orbits
- differential: counts "Floer cylinders" connecting two orbits

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Theorem (Floer 1989)

For a closed symplectic manifold* M, $HF_*(H) \cong H_{\dim(M)-*}(M)$.

Corollary (Arnold conjecture, major cases)

For a closed n-dimensional symplectic manifold* M,

$$\#\ 1\text{-periodic orbits of } X_H \geq \sum_{i=1}^{n} \operatorname{rk} H_i(M) = \sum_{i=1}^{n} b_i(M).$$

What about the double pendulum or the solar system?

Bad news I: phase spaces $\mathbb{R}^{2n}$ and $T^*\mathbb{T}^2$ are **not compact**!

Theorem (Good news for the double pendulum)

For a closed manifold M, the cotangent bundle T^*M with canonical symp. structure has symplectic homology*

$$SH_*(T^*M) \cong H_*(\Lambda M),$$

where ΛM is the free loop space of M.

The homology of $\Lambda\mathbb{T}^2$ is known.

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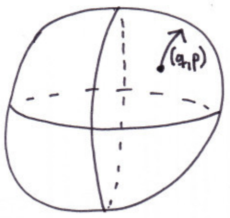
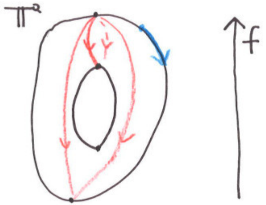
Bad news II: Hamiltonians on $\mathbb{R}^{2n}$ can have **no** periodic orbits!

Theorem (Good news for Hamiltonians on $\mathbb{R}^{2n}$)

*If $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ on $(\mathbb{R}^{2n}, \omega_0)$ has compact support, then X_H has **infinitely many** 1-periodic orbits.*

Conclusion





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- Arnold conjecture: topology of phase space forces the existence of periodic orbits.
- proof idea: gradient flow on a manifold tells you its topology.

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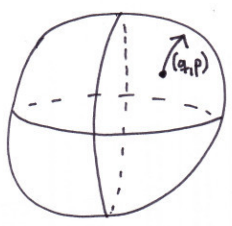

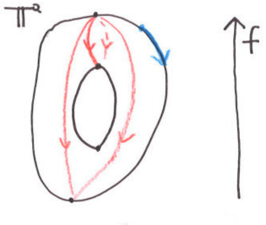

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Thanks for listening! Any questions?