Algebraic Geometry II

Exercise Sheet 5

Due Date: 26.05.2014

Exercise 1:

Let X be a k-scheme of finite type and let $k[\epsilon] = k[T]/(T^2)$ be the *ring of dual numbers* which is an infinitesimal thickening Spec $k \to$ Spec $k[\epsilon]$.

(i) Let $x \in X(k)$ be a k-valued point. Show that $\Omega^1_{X/k} \otimes \kappa(x) \cong \mathfrak{m}_x/\mathfrak{m}_x^2$, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal.

(ii) Write $f_x : \text{Spec } k \to X$ for the morphism defining x. Show that

$$\begin{aligned}\text{Def}(f_x) :&= \{f_x^{(1)} : \text{Spec } k[\epsilon] \to X \text{ morphism of } k \text{ -schemes deforming } f_x\}\\ &= \text{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x))\\ &= (\mathbf{T}_x X)(k),\end{aligned}$$

where $\mathbf{T}_x X = \text{Spec Sym}^\bullet(\mathfrak{m}_x/\mathfrak{m}_x^2)$ is the *tangent space* of X at x (viewed as a scheme).

(iii) There is a canonical closed immersion

$$\mathbf{C}_x X := \text{Spec } \Big(\bigoplus_{d\geq 0} \mathfrak{m}_x^d/\mathfrak{m}_x^{d+1}\Big) \longrightarrow \mathbf{T}_x X$$

of the *tangent cone* into the tangent space. Show further that a compatible system of deformations

$$f_x^{(n)} : \text{Spec}\, k[T]/(T^{n+1}) \to X$$

of f_x such that $f_x^{(n)}$ does not factor over Spec k gives rise to a k-valued point

$$f \in \text{Proj}\Big(\bigoplus \mathfrak{m}_x^d/\mathfrak{m}_x^{d+1}\Big)$$

or equivalently to a line in $\mathbf{C}_x X$. Deduce that $\mathbf{C}_x X \to \mathbf{T}_x X$ is an isomorphism if X is smooth at x.

(iv) Assume that X is irreducible of dimension n. Show that $\mathbf{C}_x X$ is n-dimensional. (*Hint: Show that* $\text{Bl}_{\{x\}}X$ *is n-dimensional and deduce that the fiber of* $\text{Bl}_{\{x\}}X$ *over x is $n-1$-dimensional. Then compare this fiber to* $\mathbf{C}_x X$.)

(v) Let $f : X \to Y$ be a smooth morphism. Show that the fibers of f over Y all have the same dimension, say n. Show that $\Omega^1_{X/Y}$ is locally free of rank n.

(vi) Compute the tangent space and the tangent cone of X at x in the following cases:

(a) $X = \text{Spec } k[T_1, T_2]/(T_1^3 - T_2^2)$ and $x = (0, 0)$.

(b) $X = \text{Spec } k[T_1, T_2]/(T_1^2(T_1 + 1) - T_2^2)$ and $x = (0, 0)$.

(c) $X = \text{Spec } k[T_1, T_2]/(T_1^2 + T_2^2 - 1)$ and $x = (1, 0)$.

Exercise 2:

(i) Let A be a ring and let $B = A[T_1, \dots, T_n]$. Show that $\Omega^1_{B/A} \cong \bigoplus_{i=1}^n B\,dT_i$ is free of rank n and that for $f \in A[T_1, \dots, T_n]$ one has

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial T_i} dT_i,$$

where $\partial/\partial T_i : B \to B$ is the formal derivative with respect to the variable T_i (which is a derivation).

(ii) Let A be a ring and $B = A[X, Y]/(f)$ for some $f \in A[X, Y]$. Show that

$$\Omega^1_{B,A} = (B\,dX \oplus B\,dY)/(\tfrac{\partial f}{\partial X} dX + \tfrac{\partial f}{\partial Y} dY).$$

Show that $\Omega^1_{B/A}$ is locally free of rank 1 if and only if the matrix

$$\nabla f = (\tfrac{\partial f}{\partial X}, \tfrac{\partial f}{\partial Y})$$

has rank 1 at all points of $\operatorname{Spec} B$.

Exercise 3:

Let k be a field. And let $f : \operatorname{Spec} A \to k$ be a morphism. Show that the following are equivalent:

(a) f is étale

(b) f is unramified

(c) A is isomorphic to a direct product of finitely many finite separable field extensions of k.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeomII`