Dr. E. Hellmann Wintersemester 2013/14

Exam for the lecture
Algebraic Geometry I
06.02.2014

Name	
Surname	
Matriculation number	
E-mail	

Hints:

(i) Please use a blue or black pen.

(ii) Use a new sheet for each exercise.

(iii) Throughout the whole exam k denotes an algebraically closed field.

(iv) You are allowed to use all claims and theorems from the lecture and the exercise sheets.

Exercise	1	2	3	4	5	Σ
	10	10	10	10	10	50

Grade:

Exercise 1: (5+5 points)
Let $X = \mathbb{A}_k^2 = \operatorname{Spec} k[T_1, T_2]$ and $Z = V(T_1, T_2) = \operatorname{Spec}(k[T_1, T_2]/(T_1, T_2)) \subset X$.
Let

$$f : \mathrm{Bl}_Z X = \operatorname{Proj}\big(\bigoplus_{d\geq 0} (T_1, T_2)^d\big) \longrightarrow X$$

denote the projection from the blow up of the origin to the affine plane.

(i) Show that the fiber $f^{-1}(Z)$ of f over the origin is isomorphic to $\mathbb{P}_k^1$.

(ii) Show that $\mathrm{Bl}_Z X$ is not affine.

Solution:
(i) We have

$$\begin{aligned} f^{-1}(Z) &= \operatorname{Proj}\big(\bigoplus_{d\geq 0} (T_1, T_2)^d\big) \times_X Z \\ &\cong \operatorname{Proj}\big(\bigoplus_{d\geq 0} (T_1, T_2)^d \otimes_{k[T_1,T_2]} k[T_1, T_2]/(T_1, T_2)\big) \\ &\cong \operatorname{Proj}\big(\bigoplus_{d\geq 0} (T_1, T_2)^d/(T_1, T_2)^{d+1}\big) \end{aligned}$$

Further there is an isomorphism

$$k[X_1, X_2] \longrightarrow \bigoplus_{d\geq 0} (T_1, T_2)^d/(T_1, T_2)^{d+1}$$

given by mapping X_i to T_i. Hence

$$f^{-1}(Z) \cong \operatorname{Proj}\big(\bigoplus_{d\geq 0} (T_1, T_2)^d\big) \times_X Z \cong \operatorname{Proj}(k[X_1, X_2]) = \mathbb{P}_k^1.$$

(ii) We know that a closed subscheme of an affine scheme is again affine. By (i) $\mathrm{Bl}_Z X$ contains $\mathbb{P}_k^1$ as a closed subscheme (the base change of the closed immersion $Z \hookrightarrow X$ is a closed immersion). As $\mathbb{P}_k^1$ is not affine $\mathrm{Bl}_Z X$ is not affine as well.

Exercise 2: (5+5 Points)

Show that the maps $f_k : X(k) \to Y(k)$, $(t_1, t_2, t_3) \mapsto t_3$ of k-valued points describe morphisms $f : X \to Y$ of k-schemes. Describe the fibers of these morphisms: which fibers are irreducible, which fibers are reduced?

(i) $X = \operatorname{Spec} k[T_1, T_2, T_3]/(T_1T_2 - T_3)$, $Y = \operatorname{Spec} k[T_3]$.

(ii) Assume that $\operatorname{char} k \neq 2$ and let $X = \operatorname{Spec} k[T_1, T_2, T_3]/(T_1^2 - T_2^2 + T_3^2 - 1)$, $Y = \operatorname{Spec} k[T_3]$.

Solution:

In both cases the morphism of affine schemes is induced by the map of rings

$$k[T_3] \longrightarrow k[T_1, T_2, T_3]/(T_1T_2 - T_3)$$

respectively

$$k[T_3] \longrightarrow k[T_1, T_2, T_3]/(T_1^2 - T_2^2 + T_3^2 - 1)$$

that maps T_3 to T_3.

(i) The points of Spec $k[T_3]$ are given by $\eta = (0)$ and $\xi_a = (T_3 - a)$ for $a \in k$, as k is algebraically closed. We have

$$f^{-1}(\eta) = \operatorname{Spec}(k[T_1, T_2, T_3]/(T_1T_2 - T_3) \otimes_{k[T_3]} \kappa(\eta)) = \operatorname{Spec}(k(T_3)[T_1, T_2]/(T_1T_2 - T_3))$$

and

$$f^{-1}(\xi_a) = \operatorname{Spec}(k[T_1, T_2, T_3]/(T_1T_2 - T_3) \otimes_{k[T_3]} \kappa(\xi_a)) = \operatorname{Spec}(k[T_1, T_2]/(T_1T_2 - a)).$$

If L is a field and $b \in L^\times$ we have $L[T_1, T_2]/(T_1T_2 - b) \cong L[T_1, T_1^{-1}]$ via $T_2 \mapsto b/T_1$ and this ring clearly is a domain.

Hence the fibers $f^{-1}(\eta)$ and $f^{-1}(\xi_a)$ for $a \neq 0$ are reduced and irreducible.

For $a = 0$ we have $f^{-1}(\xi_0) = \operatorname{Spec} k[T_1, T_2]/(T_1T_2)$ and the ideal (T_1T_2) is the product of the two (distinct) prime ideals (T_1) and (T_2). Hence the fiber is reduced but has two irreducible components.

(ii) With the notations form (i) we have (for the same reasons)

$$f^{-1}(\eta) = \operatorname{Spec}(k(T_3)[T_1, T_2]/((T_1 + T_2)(T_1 - T_2) + (T_3^2 - 1)))$$

and

$$f^{-1}(\xi_a) = \operatorname{Spec}(k[T_1, T_2]/((T_1 + T_2)(T_1 - T_2) + (a^2 - 1))).$$

Let L be a field with $\operatorname{char} L \neq 2$, then $L[T_1, T_2] = L[T_1 + T_2, T_1 - T_2]$ and given $b \in L^\times$ we have

$$L[T_1, T_2]/((T_1+T_2)(T_1-T_2)+b) \cong L[T_1+T_2, T_1-T_2]/((T_1+T_2)(T_1-T_2)+b) \cong L[T_1+T_2, (T_1+T_2)^{-1}]$$

for the same reasons as in (i).

Hence $f^{-1}(\eta)$ and $f^{-1}(\xi_a)$ are reduced and irreducible if $a \neq \pm 1$.

For $a = \pm 1$ we have $f^{-1}(\xi_a) \cong \operatorname{Spec}[T_1, T_2]/(T_1^2 - T_2^2)$ which is reduced and has two irreducible components, as $(T_1^2 - T_2^2)$ is the product of the two (distinct, as $\operatorname{char} k \neq 2$) prime ideals $(T_1 + T_2)$ and $(T_1 - T_2)$.

Exercise 3: (10 Points)
Let X be an irreducible topological space and $X = U_1 \cup U_2$ be a covering of X by two open subsets. Let $\mathscr{F}$ be a sheaf such that $\mathscr{F}|_{U_i}$ is the constant sheaf $\underline{A}_{U_i}$ for some abelian group A. Show that $\mathscr{F}$ is the constant sheaf $\underline{A}_X$.

Solution:
Let us write $U_{12} = U_1 \cap U_2$ and fix isomorphisms $\mathscr{F}_i = \mathscr{F}|_{U_i} \cong \underline{A}_{U_i}$. The sheaf $\mathscr{F}$ is obtained by gluing $\mathscr{F}|_{U_1}$ and $\mathscr{F}|_{U_2}$ along an isomorphism

$$\underline{A}_{U_{12}} \cong \mathscr{F}_1|_{U_{12}} = \mathscr{F}|_{U_{12}} = \mathscr{F}_2|_{U_{12}} \cong \underline{A}_{U_{12}}.$$

As U_{12} is still irreducible this isomorphism is given by a single isomorphism $\varphi : A \to A$. Let us identify $\mathscr{F}|_{U_{12}} \cong \underline{A}_{U_{12}}$ choosing the isomorphism $\mathscr{F}|_{U_{12}} = \mathscr{F}_1|_{U_{12}} \cong \underline{A}_{U_1}|_{U_{12}} = \underline{A}_{U_{12}}$. Then for $U \subset X$ we have a commutative diagram

$$\begin{array}{ccc}
\mathscr{F}(U \cap U_1) \times \mathscr{F}(U \cap U_2) & \xrightarrow{(s_1, s_2) \mapsto s_1|_{U \cap U_{12}} - s_2|_{U \cap U_{12}}} & \mathscr{F}(U \cap U_{12}) \\
\downarrow \cong & & \downarrow \cong \\
A \times A & \xrightarrow[(x_1, x_2) \mapsto x_1 - \varphi(x_2)]{} & A.
\end{array}$$

It follows that we can identify the kernel of the horizontal map with A via $x \mapsto (x, \varphi^{-1}(x))$. Which gives $\mathscr{F}(U) \cong A$.
One checks along the lines that for $V \subset U \subset X$ under the isomorphisms $\mathscr{F}(U) \cong A$ and $\mathscr{F}(V) \cong A$ just constructed the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ translates into the identity $\mathrm{id}_A : A \to A$.

Exercise 4: (3+4+3 Points)
View $X = \mathbb{A}_k^4 = \operatorname{Spec} k[T_1, T_2, T_3, T_4]$ as the scheme parametrizing all 2×2-matrices, i.e. the scheme representing the functor

$$S \longmapsto \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \Gamma(S, \mathcal{O}_S) \right\}$$

on the category of k-schemes. Let $Y \subset \mathbb{A}_k^4$ denote the functor

$$S \longmapsto \{A \in X(S) \mid A^2 = 0\}.$$

(i) Show that Y defines closed subscheme of X.

(ii) Show that the reduced subscheme underlying Y is defined by the ideal

$$(T_1 + T_4, T_1T_4 - T_2T_3) \subset k[T_1, T_2, T_3, T_4].$$

(iii) Show that Y is not reduced.

Solution:
(i) The square of the universal 2×2 matrix on X is given by

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}^2 = \begin{pmatrix} T_1^2 + T_2T_3 & T_1T_2 + T_2T_4 \\ T_1T_3 + T_3T_4 & T_2T_3 + T_4^2 \end{pmatrix}.$$

Hence Y is represented by the closed subscheme $\operatorname{Spec}(k[T_1, T_2, T_3, T_4]/I)$, where I is the ideal

$$I = (T_1^2 + T_2T_3, T_1T_2 + T_2T_4, T_1T_3 + T_3T_4, T_2T_3 + T_4^2).$$

(ii) Let $Z \subset X$ be the closed subscheme defined by the ideal $J = (T_1 + T_4, T_1T_4 - T_2T_3)$. Then Y and Z have the same underlying topological space: This may be checked on k-valued points, but a 2×2 matrix with entries in k is nilpotent if and only if its trace and its determinant vanish which is the case precisely if it defines a k-valued point of Z.
Hence we are left to show that Z is reduced. We have

$$k[T_1, T_2, T_3, T_4]/(T_1 + T_4, T_1T_4 - T_2T_3) \cong k[T_1, T_2, T_3]/(T_1^2 + T_2T_3).$$

One easily checks that $T_1^2 + T_2T_3$ is irreducible and hence the ideal generated by it is prime, as $k[T_1, T_2, T_3]$ is factorial.
(iii) It is enough to show $Y \neq Z$. But $T_1 + T_4 \in J$ and $T_1 + T_4 \notin I$.
Alternatively, let $R = k[T_1, T_2]/(T_1, T_2)^2$ and

$$A = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

be a 2×2 matrix with coefficients in R. Then $A^2 = 0$, as $T_1^2 = T_2^2 = 0$ in R and hence A defines an R-valued point of Y. However, $T_1 + T_2 \neq 0$ and hence A does not define an R-valued point of Z.

Exercise 5: (3+3+4 Points)
Let $U = \mathbb{A}^2_k \backslash \{0\}$ and write $j : U \to \mathbb{A}^2_k = X$ for the canonical embedding and $p : U \to \mathbb{P}^1_k$ for the canonical projection onto the projective line.

(i) Show that $p^*\mathcal{O}(1) \cong \mathcal{O}_U$.

(ii) Show that $j_*\mathcal{O}_U = \mathcal{O}_X$.

(iii) Let $\varphi : \mathcal{O}^2_{\mathbb{P}^1_k} \to \mathcal{O}(1)$ be the canonical surjection onto the twisting sheaf $\mathcal{O}(1)$. Show that the induced map

$$j_*p^*\varphi : \mathcal{O}^2_X \cong j_*p^*\mathcal{O}^2_{\mathbb{P}^1_k} \longrightarrow j_*p^*\mathcal{O}(1) \cong \mathcal{O}_X$$

has a non-trivial cokernel.

Solution:
(i) Let us write F for the functor that assigns to a graded $S = k[T_1, T_2]$-module a quasi-coherent sheaf on $\operatorname{Proj} k[T_1, T_2] = \mathbb{P}^1_k$. Further we write $M \mapsto \widetilde{M}$ for the functor that assigns to a (graded or not) $k[T_1, T_2]$-module M a quasi-coherent sheaf on Spec $k[T_1, T_2] = \mathbb{A}^2_k$.
We have $\mathcal{O}(1) = F(S(1))$ and

$$p^*\mathcal{O}(1) = p^*F(S(1)) \cong \widetilde{S(1)}|_U \cong \widetilde{S}|_U = \mathcal{O}_U.$$

(ii) The scheme X is noetherian and hence so is U. It follows that j_* preserves the property of being quasi-coherent. As X is affine we have

$$j_*\mathcal{O}_U \cong \Gamma(X, j_*\mathcal{O}_U)\tilde{} = \Gamma(U, \mathcal{O}_U)\tilde{}.$$

However, we computed in the lecture that $\Gamma(U, \mathcal{O}_U) = k[T_1, T_2]$ and hence $j_*\mathcal{O}_U \cong k[T_1, T_2]\tilde{} = \mathcal{O}_X$.
(iii) The map φ is defined by applying the functor F to the graded morphism of graded rings

$$\Phi : S^2 = Se_1 \oplus Se_2 \longrightarrow S(1)$$

mapping e_i to $T_i \in S_1 = S(1)_0$. It follows that $p^*\varphi$ is the restriction of $\tilde{\Phi}$ to U and $j_*p^*\varphi$ is determined by its effect on global sections. However the morphism

$$p^*\varphi = \tilde{\Phi}|_U : \widetilde{S^2}|_U \longrightarrow \widetilde{S(1)}|_U$$

gives on global sections just Φ itself which has a non trivial cokernel (given by k in homogenous degree -1). Hence

$$\operatorname{coker} j_*p^*\varphi = (\operatorname{coker} \Phi)\tilde{} \neq 0.$$

Alternative solution (for (i) and (iii))
We define a surjection $\psi : \mathcal{O}^2_U = \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2 \to \mathcal{O}_U$ by setting $\psi(e_i) = T_i$. Then the map $p : U \to \mathbb{P}^1_k$ is just the map defined by ψ using the functorial description of $\mathbb{P}^1_k$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}^2_U & \xrightarrow{\psi} & \mathcal{O}_u \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \cong} \\ p^*\mathcal{O}_{\mathbb{P}^1_k} & \xrightarrow[p^*\varphi]{} & p^*\mathcal{O}(1). \end{array}$$

Especially $p^*\mathcal{O}(1) \cong \mathcal{O}_U$ proving (i). Further, on global sections ψ is the map Φ from above. Hence $j_*p^*\varphi$ is given by $\tilde{\Phi}$ and we conclude as above.