Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 9

Due Date: 19.12.2013

Exercise 1:

Let $f : X \to Y$ and $g : y \to Z$ and $h : Y' \to Y$ be morphisms of schemes. Let $\mathcal{P}$ be one of the properties being *locally of finite type*, resp. *locally of finite presentation*, resp. *quasi-compact*.

(i) Show that if f and g have the property $\mathcal{P}$, then $g \circ f$ has the property $\mathcal{P}$ as well.

(ii) Let f be a closed immersion. Show that f is of finite type. Show that f is of finite presentation if Y is locally noetherian.

(iii) Let X' denote the fiber product $X' = X \times_Y Y'$ and write $f' : X' \to Y'$ for the canonical projection. Show that f' has the property $\mathcal{P}$ if f has the property $\mathcal{P}$.

Exercise 2:

(i) Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring. Show that

(a) $\operatorname{Proj} S$ is empty if and only if every element of S_+ is nilpotent.

(b) $\operatorname{Proj} S$ is reduced if S is reduced.

(c) $\operatorname{Proj} S$ is integral if S has no zero-divisors.

(ii) Let k be a field and let $\bar{k}$ be an algebraic closure, k_s be a separable closure and k_p be a perfect closure of k. Let X be a k-scheme. Show that

(a) $X \times_k \bar{k}$ is irreducible if and only if $X \times_k k_s$ is irreducible if and only if $X \times_k K$ is irreducible for all field extension K of k.

(b) $X \times_k \bar{k}$ is reduced if and only if $X \times_k k_p$ is reduced if and only if $X \times_k K$ is reduced for all field extension K of k.

Exercise 3:

(i) Let $\varphi : S \to T$ be a graded morphisms of graded rings and let $U = \{\mathfrak{p} \in \operatorname{Proj} T \mid \mathfrak{p} \not\supset \varphi(S_+)\} \subset \operatorname{Proj} T$. Show that $U \subset \operatorname{Proj} T$ is open and that φ defines a natural morphism $f : U \to \operatorname{Proj} S$.

(ii) Assume that φ is surjective. Show that $U = \operatorname{Proj} T$ and that f is a closed immersion.

(iii) Assume that S is generated by finitely many homogeneous elements of degree 1 and write $A = S_0$. Show that the canonical morphism $\operatorname{Proj} S \to \operatorname{Spec} A$ is projective, i.e. that there exists some $N \geq 0$ and a commutative diagram

$$\begin{array}{ccc} \operatorname{Proj} T & \xrightarrow{f} & \mathbb{P}^N_A \\ & \searrow & \downarrow \\ & & \operatorname{Spec} A \end{array}$$

such that f is a closed immersion.

Exercise 4:

(i) Let $X = \mathrm{Spec}(k[T_1, T_2]/(T_2^2 - T_1^2(T_1 + 1))) \to \mathrm{Spec}(k[T_1, T_2]) = \mathbb{A}_k^2$ and let $Z \subset X \subset \mathbb{A}_k^2$ be the closed subscheme consisting of the origin. Show that the Blow-up $\mathrm{Bl}_X Z$ of X along Z embeds into $\mathrm{Bl}_{\mathbb{A}_k^2} Z$ and hence into $\mathbb{A}_k^2 \times \mathbb{P}_k^1$.

(ii) Show that $\mathbb{A}_k^1 \cong \mathrm{Bl_X Z}$ and that in (homogeneous) coordinates the embedding of $\mathbb{A}_k^1 \hookrightarrow \mathbb{A}_k^2 \times \mathbb{P}^1$ is under this isomorphisms given by $t \mapsto (t^2 - 1, t(t^2 - 1), t)$.

(iii) What happens if we blow up the origin on $Y = \mathrm{Spec}(k[T_1, T_2]/(t_2^2 - T_1^3)) \hookrightarrow \mathbb{A}_k^2$?

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`