Dr. E. Hellmann WS 2013/14

Algebraic Geometry I

Exercise Sheet 5

Due Date: 21.11.2013

Exercise 1:

Let X be a prevariety and let $Z \subset X$ be an irreducible closed subset. Let us write $\iota : Z \hookrightarrow X$ for the inclusion of Z.

(i) Let $\mathscr{J} \subset \mathcal{O}_X$ denote the presheaf $U \mapsto \{f \in \mathcal{O}_X(U) \mid f|_{U\cap Z} = 0\}$. Show that $\mathscr{J}$ is a sheaf and that the quotient $\mathcal{O}_X/\mathscr{J}$ in the category of sheaves equals the sheaf $\iota_*\mathcal{O}_Z$.

(ii) Let $X = \mathbb{P}^1$ and write $\iota_x : \{x\} \hookrightarrow \mathbb{P}^1$ for the inclusion of a point $x \in \mathbb{P}^1$. Let $\mathcal{K}$ denote the constant sheaf $\underline{K}$ on $\mathbb{P}^1$, where $K = K(\mathbb{P}^1)$ is the function field of $\mathbb{P}^1$. Show that there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{x\in\mathbb{P}^1} \iota_*(K/\mathcal{O}_{\mathbb{P}^1,x}) \longrightarrow 0.$$

Exercise 2:

(i) Let X be a topological space and let $U_i \subset X$, $i \in I$ be open subsets such that $X = \bigcup_{i\in I} U_i$. For $i \in I$ let $\mathscr{F}_i$ be a sheaf on U_i. Assume that for each pair (i, j) there are isomorphisms $\varphi_{ij} : \mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}_j|_{U_i\cap U_j}$ satisfying the *cocycle condition*, i.e. for all indices i, j, k one has $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Show that there exists a sheaf $\mathscr{F}$ and isomorphisms $\psi_i : \mathscr{F}_i \to \mathscr{F}|_{U_i}$ such that $\psi_j \circ \varphi_{ij} = \psi_i$ on $U_i \cap U_j$. Show further that $(\mathscr{F}, \psi_i)$ is uniquely determined up to unique isomorphism by these conditions.

(ii) Let X be a topological space and let $\mathcal{B}$ be a basis of the topology. For each $U \in \mathcal{B}$ we give an abelian group $\mathscr{F}(U)$ and for $U, V \in \mathcal{B}$ such that $U \subset V$ we give restriction maps $\mathrm{res}^V_U : \mathscr{F}(V) \to \mathscr{F}(U)$ such that $\mathrm{res}^U_U = \mathrm{id}_{\mathscr{F}(U)}$ and $\mathrm{res}^W_U = \mathrm{res}^V_U \circ \mathrm{res}^W_V$ for all $U, V, W \in \mathcal{B}$ satisfying $U \subset V \subset W$. Further we assume that for each $U \in \mathcal{B}$ and any open covering $U = \bigcup_{i\in I} U_i$ of U by elements $U_i \in \mathcal{B}$ the sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\ \phi\ } \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\ \psi\ } \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$$

is exact, where maps are given by

$$\phi : s \longmapsto (\mathrm{res}^U_{U_i} s)_i$$
$$\psi : (s_i)_i \longmapsto (\mathrm{res}^{U_i}_{U_i\cap U_j} s_i - \mathrm{res}^{U_j}_{U_i\cap U_j} s_j)_{ij}.$$

Show that there is a unique sheaf $\mathscr{F}$ on X that agrees on elements of $\mathcal{B}$ with $(\mathscr{F}(U), \mathrm{res}^V_U)_{U,V\in\mathcal{B}}$.

Exercise 3:
Let X be a topological space.

(i) Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of abelian groups on X. For $U \subset X$ open let $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ denote the group of homomorphisms of the restricted sheaves. Show that $U \mapsto \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ defines a sheaf, denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

(ii) Let $(\mathscr{F}_i, f_{ij})$ be a projective system of sheaves of abelian groups on X. Show that the presheaf

$$U \longmapsto \varprojlim \mathscr{F}_i(U)$$

is a sheaf and is the inverse limit of the system $(\mathscr{F}_i, f_{ij})$ in the category of sheaves of abelian groups on X.

(iii) Let $(\mathscr{F}_i, f_{ij})$ be an inductive system of sheaves of abelian groups on X. Show that the sheafification of the presheaf

$$U \longmapsto \varinjlim \mathscr{F}_i(U)$$

is the colimit of the inductive system $(\mathscr{F}_i, f_{ij})$ in the category of sheaves of abelian groups on X.

Exercise 4:
Let $\mathscr{C}$ and $\mathscr{D}$ be categories and let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. The functors F and G are called a pair of *adjoint functors* if for all objects $A \in \mathscr{C}$ and $B \in \mathscr{D}$ there exists isomorphisms

$$\mathrm{Hom}_{\mathscr{D}}(F(A), B) \cong \mathrm{Hom}_{\mathscr{C}}(A, G(B))$$

which are functorial in A and B. More precisely one says that G is *right adjoint* to F and F is *left adjoint* to G.

(i) Let $(M_i)_{i \in I}$ be an inductive system in $\mathscr{C}$ such that $M = \varinjlim M_i$ exists in $\mathscr{C}$. Show that $F(M)$ is the inductive limit of the system $(F(M_i))_{i \in I}$, i.e. F commutes with colimits. Show that G commutes with limits.

(ii) Assume that $\mathscr{C}$ and $\mathscr{D}$ are abelian categories[1] Show that F is right exact and G is left exact.

(iii) Let A be a ring and M be an A-module. Show that the functors

$$N \longmapsto \mathrm{Hom}_A(M, N)$$
$$N \longmapsto N \otimes_A M$$

form a pair of adjoint functors.

(iv) Let X be a topological space. Show that the sheafification $\mathscr{F} \mapsto \mathscr{F}^+$ from the category PreSh_X of presheaves (of abelian groups) on X to the category Sh_X of sheaves (of abelian groups) on X is left adjoint to the forgetful functor $\iota : \mathrm{Sh}_X \to \mathrm{PreSh}_X$.

Homepage: `www.math.uni-bonn.de/people/hellmann/alggeom`

[1] If you are not familiar with the notion of an abelian category, you can assume that $\mathscr{C}$ is the category of A-modules and $\mathscr{D}$ is the category of B-modules for some rings A and B.