1. Suppose that $f : M \to \mathbb{R}$ has a critical point at $p \in M$. Given coordinates $(x_1, \ldots, x_n)$ around p, we consider the Hessian matrix

$$(\partial_{x_i}\partial_{x_j} f)(p) \tag{0.1}$$

Check that the following properties are independent of the choice of coordinates:

- the Hessian is nonsingular
- the Hessian has d negative eigenvalues (counted with multiplicity).

Check also that you can always choose coordinates so that the eigenvalues are all ± 1.

2. Prove that a Morse function on a compact manifold has only finitely many critical points.

3. Suppose that (f, g) is a Morse–Smale pair and let $p, q \in \mathrm{crit}(f)$. If $\mathcal{M}(p, q) \neq \emptyset$, then $|q| < |p|$.

4. Let M, N be closed manifolds. Prove that

$$HM_*(M \times N; \mathbb{Z}/2) \simeq HM_*(M; \mathbb{Z}/2) \otimes HM_*(N; \mathbb{Z}/2). \tag{0.2}$$

5. The 2-torus does not admit a Morse function with three or fewer critical points. *Remark: it is natural to wonder what happens if we drop the qualifier "Morse". In fact, the minimal number of critical points of* any *smooth function on the* 2*-torus is* 3*. This can be proved using Lusternik–Schnirelmann theory.*

6. Let f be a Morse function and fix a metric g. If p is a critical point of index k, the unstable manifold $W^u(p)$ is diffeomorphic to D^k (a k-dimensional open ball). *Remark: you may use the Morse lemma, which was stated but not proved in class.*

7. Let R be a commutative ring. Suppose that M is a closed n-dimensional manifold endowed with an R-orientation (i.e. a distinguished isomorphism of R-modules $R \simeq H_n(M; R)$). Explain how to construct coherent orientations of the compactified moduli spaces $\overline{\mathcal{M}}(p, q)$, for all $p, q \in \mathrm{crit}(f)$ with $|p| - 2 < |q| < |p|$. *Hint: choose orientations arbitrarily on all unstable manifolds.*