Fundamental Lemma and Hitchin Fibration

Gérard Laumon

CNRS and Université Paris-Sud

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In order to:

- compute the Hasse-Weil zeta functions of Shimura varieties (for example $\mathcal{A}_g$),
- prove endoscopic cases of the Langlands functoriality (for example the transfer from $\mathrm{Sp}(2n)$ to $\mathrm{GL}(2n)$),

one first needs to stabilize the Arthur-Selberg trace formula.

This stabilization can only be done after having established some combinatorial identities between orbital integrals for p-adic reductive groups.

The series of these conjectural identities form the so-called "Fundamental Lemma".

- There are four variants of the Fundamental Lemma: ordinary, twisted, weighted and twisted weighted.

- Here we only consider the ordinary Fundamental Lemma.

- First occurrence of the Fundamental Lemma in Labesse-Langlands' paper (1979).

- General formulation of the Fundamental Lemma by Langlands-Shelstad (1987).

ORBITAL INTEGRALS FOR $\mathrm{GL}(n)$

$$\mathrm{O}_\gamma^G = \int_{G_\gamma(F)\backslash G(F)} 1_{\mathcal{K}}(g^{-1}\gamma g)\frac{\mathrm{d}g}{\mathrm{d}g_\gamma}$$

F non archimedan local field: for example $F = \mathbb{Q}_p$ or $\mathbb{F}_p((\varpi))$
$\mathcal{O}_F$ the ring of integers of F: $\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p$ and $\mathcal{O}_{\mathbb{F}_p((\varpi))} = \mathbb{F}_p[[\varpi]]$

$G = \mathrm{GL}(n)$, $G(F)$ p-adic or ϖ-adic Lie group
$K = G(\mathcal{O}_F) \subset G(F)$ maximal compact open subgroup

$\mathfrak{g} = \mathrm{gl}(n, F)$ the Lie algebra of $G(F)$
$\mathcal{K} = \mathrm{Lie}(K) = \mathrm{gl}(n, \mathcal{O}_F)$: an $\mathcal{O}_F$-lattice in the F-vector space $\mathfrak{g}$
$1_{\mathcal{K}} : \mathfrak{g} \to \{0, 1\}$ the characteristic function of $\mathcal{K}$

$\gamma \in \mathfrak{g}$ regular semisimple
$\Rightarrow$ its centralizer G_γ is a maximal torus in G

$\mathrm{d}g$ and $\mathrm{d}g_\gamma$ Haar measures on $G(F)$ and $G_\gamma(F)$

ORBITAL INTEGRALS FOR $\mathrm{GL}(n)$ AS NUMBERS OF LATTICES

Lemma

$$\mathrm{O}_\gamma^G = |X_\gamma/\Lambda_\gamma|.$$

Here:

- $X_\gamma = \{\mathcal{O}_F\text{-lattices } M \subset F^n \mid \gamma(M) \subset M\}$,
- $\gamma \in \mathfrak{g}$ regular semisimple $\Leftrightarrow$ $F[\gamma] \subset \mathfrak{g}$ commutative semisimple F-algebra of dimension $n \Rightarrow F[\gamma] = \prod_{i\in I} E_i$ where $(E_i)_{i\in I}$ is a finite family of finite separable extensions of F,
- choices of uniformizers ϖ_{E_i}'s in the E_i's $\Rightarrow F[\gamma]^\times \cong \Lambda_\gamma \times K_\gamma$ where $\Lambda_\gamma = \mathbb{Z}^I$ and $K_\gamma = \prod_{i\in I} \mathcal{O}_{E_i}^\times$ maximal compact open subgroup of $G_\gamma(F) = F[\gamma]^\times$,
- $\Lambda_\gamma \subset G_\gamma(F)$ acts freely on X_γ,
- $\mathrm{vol}(K, \mathrm{d}g) = \mathrm{vol}(K_\gamma, \mathrm{d}g_\gamma) = 1$.

UNITARY GROUPS

- $[F' : F] = 2$ unramified, $\mathrm{Gal}(F'/F) = \{1, \tau\}$.
- $\Phi_n : F'^n \times F'^n \to F',\ (x, y) \mapsto x_1^\tau y_n + x_2^\tau y_{n-1} + \cdots + x_n^\tau y_1$, standard hermitian form
  $\Rightarrow G(F) = \mathrm{U}(n, F) \subset \mathrm{GL}(n, F')$ unitary group.
- $(E_i)_{i \in I}$ finite family of finite separable extensions of F such that E_i is disjoint of F'
  $\Rightarrow E_i' = E_i F'$, $\mathrm{Gal}(E_i'/E_i) \cong \mathrm{Gal}(F'/F) = \{1, \tau\}$.
- $c = (c_i)_{i \in I}$, $c_i \in E_i \Rightarrow (E_I', \Phi_{I,c}) = \bigoplus_{i \in I} (E_i', \Phi_{i,c_i})$ hermitian space where: $\Phi_{i,c_i} : E_i' \times E_i' \to F',\ (x, y) \mapsto \mathrm{Tr}_{E_i'/F'}(c_i x^\tau y)$.
- $\mathrm{discr}(\Phi_{i,c_i}), \mathrm{discr}(\Phi_{I,c}) \in F^\times / \mathrm{N}_{F'/F} F'^\times \cong \mathbb{Z}/2\mathbb{Z}$,

$$\mathrm{discr}(\Phi_{I,c}) = \sum_{i \in I} \mathrm{discr}(\Phi_{i,c_i}).$$

ORBITAL INTEGRALS FOR UNITARY GROUPS

- Assume $\mathrm{discr}(\Phi_{I,c}) = 0 \Rightarrow (E'_I, \Phi_{I,c}) \cong (F'^n, \Phi_n)$.
- Choosing such an isomorphism
  $\Rightarrow \iota_c : \bigoplus_{i \in I} \{x \in E'_i \mid x^\tau + x = 0\} \subset \mathrm{End}_{F'}(E'_I, \Phi_{I,c}) \cong \mathfrak{g}$.
- $\gamma = (\gamma_i)_{i \in I} \in E'_I$ such that:
  - $\gamma_i^\tau + \gamma_i = 0$,
  - $E'_i = F'[\gamma_i] \cong F'[x]/(P_i)$, P_i the minimal polynomial of γ_i,
  - $(P_i, P_j) = 1$, $\forall\ i \neq j$,

  $\Rightarrow$ regular semisimple $\gamma_c = \iota_c(\gamma) \in \mathfrak{g}$.

Lemma

$\mathrm{O}^G_{\gamma_c} =$ *the number of $\mathcal{O}_{F'}$-lattices $M \subset E'_I$ such that:*

- $M^{\perp_c} := \{x \in E'_I \mid \Phi_{I,c}(x, M) \subset \mathcal{O}_{F'}\} = M$,
- $\gamma M \subset M$.

STABLE CONJUGACY

- The $G(F)$-conjugacy class of γ_c in $\mathfrak{g}$ does not depend on the choice of the isomorphism $(E_I', \Phi_{I,c}) \cong (F'^n, \Phi_n)$.

- The γ_c's are stably conjugated: they are conjugated in $\mathrm{GL}(n, F')$ but not necessarily in $G(F)$.

- The $G(F)$-conjugacy class of γ_c only depends on

$$\mu(c) = (\mathrm{discr}(\Phi_{i,c_i}))_{i \in I} \in (\mathbb{Z}/2\mathbb{Z})^I.$$

- As $\mathrm{discr}(\Phi_{I,c}) = 0$, $\mu(c)$ lives in $\Lambda_\gamma^0 / 2\Lambda_\gamma^0$ where $\Lambda_\gamma^0 = \mathrm{Ker}(+ : \mathbb{Z}^I \to \mathbb{Z})$.

In other words γ defines a stable conjugacy class in $\mathfrak{g}$ and inside this stable conjugacy class there are finitely many $G(F)$-conjugacy classes, which are parametrized by $\Lambda_\gamma^0 / 2\Lambda_\gamma^0$.

κ-ORBITAL INTEGRALS FOR UNITARY GROUPS

- For each $\mu \in \Lambda_\gamma^0/2\Lambda_\gamma^0$ let us choose c_μ with $\mu(c_\mu) = \mu$. The γ_{c_μ}'s form a system of representatives of the $G(F)$-conjugacy classes in the stable conjugacy class defined by γ.

- For any $\kappa : \Lambda_\gamma^0/2\Lambda_\gamma^0 \to \{\pm 1\}$ we then have the κ-orbital integral:
$$\mathrm{O}_\gamma^{G,\kappa} = \sum_{\mu \in \Lambda_\gamma^0/2\Lambda_\gamma^0} \kappa(\mu)\mathrm{O}_{\gamma_{c_\mu}}^G.$$

- For $\kappa = 1$, the κ-orbital integral is also called the stable orbital integral:
$$\mathrm{SO}_\gamma^G := \mathrm{O}_\gamma^{G,1}.$$

LANGLANDS-SHELSTAD FUNDAMENTAL LEMMA FOR UNITARY GROUPS

Conjecture (Langlands-Shelstad)

$$\mathrm{O}_\gamma^{G,\kappa} = (-q)^r \mathrm{SO}_\gamma^H.$$

Here:

- $G = \mathrm{U}(n)$, $\gamma = \gamma_I = (\gamma_i)_{i \in I} \in \bigoplus_{i \in I} \{x \in E_i' \mid x^\tau + x = 0\}$,
- $\kappa : \Lambda_\gamma^0 / 2\Lambda_\gamma^0 \to \{\pm 1\} \Leftrightarrow I = I_1 \amalg I_2$
  $\Rightarrow$ endoscopic group $H = \mathrm{U}(n_1) \times \mathrm{U}(n_2)$ with $n_\alpha = |I_\alpha|$,
- $\mathrm{SO}_\gamma^H := \mathrm{SO}_{\gamma_{I_1}}^{\mathrm{U}(n_1)} \times \mathrm{SO}_{\gamma_{I_2}}^{\mathrm{U}(n_2)}$,
- q the number of elements of the residue field of F,
- r the valuation of the resultant of $\prod_{i \in I_1} P_i$ and $\prod_{i \in I_2} P_i$ (P_i the minimal polynomial of γ_i over F').

The computation of the transfer factor is due to Waldspurger.

RESULTS (CLASSICAL METHODS)

- Labesse-Langlands (1979): $\mathrm{U}(2)$.
- Kottwitz (1992) and Rogawski (1990): $\mathrm{U}(3)$.
- Waldspurger (2005):
  The equal characteristic case ($F \supset \mathbb{F}_p((\varpi))$)
  $\Rightarrow$ the unequal characteristic case ($F \supset \mathbb{Q}_p$).

RESULTS (GEOMETRIC METHODS)

$F = \mathbb{F}_q((\varpi))$, $\mathbb{F}_q$ finite field of characteristic p.

Theorem (Goresky-Kottwitz-MacPherson)

The Langlands-Sheldstad Fundamental Lemma for unitary groups holds if the following conditions are satisfied

- $p \gg 0$,
- $E_i = E$ *does not depend on* $i \in I$ *and is unramified over* F,
- $v_F(\alpha(\gamma)) = v_F(\beta(\gamma))$ *for every pair of roots* (α, β).

This is the unramified equal valuation case.

Theorem (Ngô-L.)

The Langlands-Sheldstad Fundamental Lemma for unitary groups holds if $p > n$.

Grothendieck-Lefschetz fixed point formula

The key of the geometric approaches is:

Theorem (Grothendieck-Lefschetz fixed point formula)

$$\mathrm{O}_\gamma^{G,\kappa} = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Frob}_q^*, H^i(X_\gamma^0/\Lambda_\gamma^0, \mathcal{L}_\kappa)).$$

Here:

- X_γ^0 is an algebraic variety over $\overline{\mathbb{F}}_q$, a connected component of the affine Springer fiber X_γ,
- Λ_γ^0 is a lattice acting freely on X_γ^0,
- $\mathcal{L}_\kappa$ is the rank 1 ℓ-adic local system on $X_\gamma^0/\Lambda_\gamma^0$ defined by the covering $X_\gamma^0 \to X_\gamma^0/\Lambda_\gamma^0$ and the character κ of its Galois group Λ_γ^0 ($\ell \neq \mathrm{Char}(\mathbb{F}_q)$),
- Frob_q is a suitable Frobenius endomorphism.

Natural expectation: The Fundamental Lemma is the consequence of a (stronger) cohomological statement.

AFFINE SPRINGER FIBERS FOR $\mathrm{GL}(n)$

k algebraically closed, $n > 0$

$\Rightarrow$ the affine Grassmannian: $X = \{k[[\varpi]]\text{-lattices } M \subset k((\varpi))^n\}$, an ind-scheme over k whose connected components are: $X^d = \{M \in X \mid [M : k[[\varpi]]^n] = d\},\ d \in \mathbb{Z}$.

$\gamma \in \mathrm{gl}(n, k((\varpi)))$ regular semisimple

$\Rightarrow$ the affine Springer fiber: $X_\gamma = \{M \in X \mid \gamma(M) \subset M\}$, a closed ind–subscheme of X.

$k((\varpi))[\gamma] = \prod_{i \in I} E_i$, choosing uniformizers ϖ_{E_i}'s of E_i's

$\Rightarrow$ free action of $\Lambda_\gamma = \mathbb{Z}^I$ on X_γ by $\lambda \cdot M = (\varpi_{E_i}^{-\lambda_i})_{i \in I}(M)$, $\Lambda_\gamma^0 = \mathrm{Ker}(+ : \mathbb{Z}^I \to \mathbb{Z})$ stabilizes X_γ^0.

Theorem (Kazhdan-Lusztig)

- *X_γ scheme locally of finite type over k and of finite dimension, whose connected components are the $X_\gamma^d := X_\gamma \cap X^d$'s,*
- *$X_\gamma / \Lambda_\gamma = X_\gamma^0 / \Lambda_\gamma^0$ is a projective scheme.*

FROBENIUS ENDOMORPHISM

$k = \overline{\mathbb{F}}_q$

Twisted Frobenius on $\mathrm{GL}(n, k((\varpi)))$ with respect to $\mathbb{F}_q$:

$$\mathrm{Frob}_q(g) = \Phi \cdot {}^{\mathrm{t}}\Big(\sum_m g_{ij,m}^q \varpi^m\Big)^{-1} \cdot \Phi, \quad \Phi = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$$

$\Rightarrow$ $\mathrm{U}(n, \mathbb{F}_q((\varpi)))$.

γ regular semisimple in $\mathrm{gl}(n, k((\varpi)))$

$\Rightarrow$ affine Springer fiber X_γ and its quotient $X_\gamma/\Lambda_\gamma = X_\gamma^0/\Lambda_\gamma^0$.

γ fixed by Frob_q

$\Rightarrow$ a twisted Frobenius endomorphism Frob_q on $X_\gamma^0/\Lambda_\gamma^0$.

GORESKY-KOTTWITZ-MACPHERSON APPROACH

If a projective variety over $k = \overline{\mathbb{F}}_q$ is equipped with a torus action satisfying the following properties:

- the fixed point set is finite,
- the set of one-dimentional orbits is finite,
- the ordinary ℓ-adic cohomology is pure,

then one can explicitely compute its ℓ-adic cohomology:

- one first computes its ℓ-adic equivariant cohomology for the torus action by using Atiyah-Borel-Segal's localization to the fixed point set,
- one recovers the ordinary cohomology from the equivariant one.

PURE ℓ-ADIC COHOMOLOGY

$k = \overline{\mathbb{F}}_q$, $\ell \neq p$, Z separated scheme of finite type over k

- Z defined over $\mathbb{F}_q \Rightarrow \mathrm{Frob}_q$ acts on $H^i(Z, \mathbb{Q}_\ell)$,
- $H^i(Z, \mathbb{Q}_\ell)$ is pure of weight i
  $\Leftrightarrow \forall$ eigenvalue α of Frob_q, $|\alpha| = q^{\frac{i}{2}}$,
- Z has pure ℓ-adic cohomology
  $\Leftrightarrow H^i(Z, \mathbb{Q}_\ell)$ is pure of weight i, $\forall\ i$.

Theorem (Deligne's main theorem)

Assume Z proper and smooth over k. Then Z has pure ℓ-adic cohomology.

ℓ-ADIC COHOMOLOGY OF AFINE SPRINGER FIBERS

The affine Springer fiber X_γ not of finite type but $H^i(X_\gamma, \mathbb{Q}_\ell)$ makes sense.

Assume that G and γ are defined over $\mathbb{F}_q$.

Conjecture (Goresky-Kottwitz-MacPherson)

The ℓ-adic cohomology of X_γ is pure.

Theorem (Goresky-Kottwitz-MacPherson)

Assume that γ is of equal valuations. Then the ℓ-adic cohomology of X_γ is pure.

TORUS ACTIONS ON AFFINE SPRINGER FIBERS

$k = \overline{\mathbb{F}}_q$, $T \subset G = \mathrm{GL}(n)$ maximal torus of diagonal matrices,

$\gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n) \in \mathrm{gl}(n, k[[\varpi]])$ regular semisimple ($\gamma_i \neq \gamma_j$, $\forall\ i \neq j$)

$\Rightarrow$ $X_\gamma = \{M \subset k((\varpi))^n \mid \gamma M \subset M\}$,

T and $\Lambda = X_*(T) = \mathbb{Z}^n$ act on X_γ and the two actions commute.

The fixed point set is discrete:

- $X_\gamma^T = \{\bigoplus_{i=1}^n \varpi^{-\lambda_i} k[[\varpi]] \mid \lambda \in \Lambda\}$,
- X_γ^T plus action of $\Lambda \cong \Lambda$ plus action by translations on itself.

EQUIVARIANT COHOMOLOGY OF AFFINE SPRINGER FIBERS

$H^\bullet_T(\mathrm{Spec}(k), \mathbb{Q}_\ell) = \mathrm{Sym}^\bullet X^*(T) \otimes \mathbb{Q}_\ell \supset \mathfrak{a} = H^{\bullet>0}_T(\mathrm{Spec}(k), \mathbb{Q}_\ell)$,

$H^\bullet_T(X^T_\gamma, \mathbb{Q}_\ell) = \mathrm{Sym}^\bullet X^*(T) \otimes \mathbb{Q}_\ell[[\Lambda]]$.

Theorem (Goresky-Kottwitz-MacPherson)

Assume that $H^\bullet(X_\gamma, \mathbb{Q}_\ell)$ is pure. Then:

- *the restriction map $H^\bullet_T(X_\gamma, \mathbb{Q}_\ell) \to H^\bullet_T(X^T_\gamma, \mathbb{Q}_\ell)$ is injective,*
- *its image = set of $f \in \mathrm{Sym}^\bullet X^*(T) \otimes \mathbb{Q}_\ell[[\Lambda]]$ such that*

$$f(1 - \alpha^\vee)^d \in \alpha^d \mathrm{Sym}^{\bullet - d} X^*(T) \otimes \mathbb{Q}_\ell[[\Lambda]],$$

  $\forall\ \alpha \in R(G, T)$, $\forall\ d = 1, 2, \ldots, v_F(\alpha(\gamma))$,
- $H^\bullet(X_\gamma, \mathbb{Q}_\ell) = H^\bullet_T(X_\gamma, \mathbb{Q}_\ell) / \mathfrak{a} H^\bullet_T(X_\gamma, \mathbb{Q}_\ell)$.

NGÔ-L. APPROACH

First main idea: Deform complicated affine Springer fibers into simpler ones (look for an analog of Grothendieck-Springer simultaneous resolution of the nilpotent cone).
Problem: It does not seem to work!

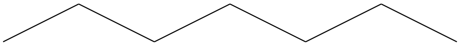
Second main idea: Replace affine Springer fibers (local objects) by compactified Jacobians (global objects).

Third main idea: Hitchin fibration is a wonderful group theoretical family of compactified Jacobians.

PROBLEM IN DEFORMING AFFINE SPRINGER FIBERS

$$\gamma_t = \begin{pmatrix} t\varpi & 1 \\ \varpi^3 & -t\varpi \end{pmatrix} \in \mathrm{gl}(2, k[[\varpi]])$$

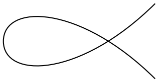
For $t \neq 0$ the affine Springer fiber is a chain of projective lines



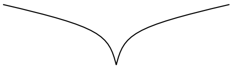
and for $t = 0$ it is a single projective line

$\Rightarrow$ no algebraic family.

Replace the affine Springer fiber at $t \neq 0$ by



and at $t = 0$ by



$\Rightarrow$ a nice algebraic family.

TORSION FREE MODULES

$\gamma \in \mathrm{gl}(n, k[[\varpi]]) \subset \mathrm{gl}(n, k((\varpi)))$ regular semisimple,
$P = P(\varpi, x) \in k[[\varpi]][x]$ minimal polynomial of γ,
$R = k[[\varpi]][\gamma] = k[[\varpi]][x]/(P) \subset \mathrm{Frac}(R) = k((\varpi))[x]/(P)$,
$\mathrm{Spf}(R)$ formal germ of plane curve $(P(\varpi, x) = 0)$.

- P_R moduli space of invertible R-modules M equipped with a rigidification $M \otimes_R \mathrm{Frac}(R) \cong \mathrm{Frac}(R)$, a commutative group scheme over k called the local Jacobian of $\mathrm{Spf}(R)$,
- $\overline{P}_R$ moduli space of torsion free R-modules M equipped with a rigidification $M \otimes_R \mathrm{Frac}(R) \cong \mathrm{Frac}(R)$, an equivariant compactification of P_R called the local compactified Jacobian of $\mathrm{Spf}(R)$.

Proposition

The tautological map $X_\gamma \to \overline{P}_R$, ($M$ stable by γ) $\mapsto$ (the R-module M), is an homeomorphism.

COMPACTIFIED JACOBIANS

C integral projective curve over k with only plane curve singularities ($\hat{\mathcal{O}}_{C,c} \cong k[[x,y]]/(f)$, $\forall c \in C$).

- $\mathrm{Pic}(C)$ the moduli space of locally free $\mathcal{O}_C$-Modules of rank 1, a commutative group scheme over k called the Picard scheme or Jacobian of C,
- $\overline{\mathrm{Pic}}(C)$ the moduli space of torsion free $\mathcal{O}_C$-Modules of generic rank 1, an equivariant compactification of $\mathrm{Pic}(C)$ called the compactified Jacobian of C.

Proposition

We have a natural isomorphism of algebraic stacks:

$$[\overline{\mathrm{Pic}}(C)/\mathrm{Pic}(C)] \cong \prod_{c \in C} [\overline{P}_{\hat{\mathcal{O}}_{C,c}}/P_{\hat{\mathcal{O}}_{C,c}}].$$

PURITY CONJECTURE FOR COMPACTIFIED JACOBIANS

The purity conjecture of Goresky, Kottwitz and MacPherson for affine Springer fibers together with the previous two propositions ($X_\gamma \cong \overline{P}_R$ and $[\overline{\mathrm{Pic}}(C)/\mathrm{Pic}(C)] \cong \prod_{c \in C} [\overline{P}_{\hat{\mathcal{O}}_{C,c}}/P_{\hat{\mathcal{O}}_{C,c}}]$) imply:

Conjecture

Let C be any integral projective curve over $k = \overline{\mathbb{F}}_q$. Assume C has only unibranch plane curve singularities. Then the ℓ-adic cohomology of $\overline{\mathrm{Pic}}(C)$ is pure.

Variant with no unibranch assumption: replace $\overline{\mathrm{Pic}}(C)$ by a suitable étale covering.

HITCHIN FIBRATION

Γ a fixed connected smooth projective curve over k, $g(\Gamma) \geq 2$,

Δ a given effective Cartier divisor on Γ, $\deg(\Delta) > 2g(\Gamma) - 2$,

$\mathcal{M}$ the algebraic stack of Higgs (or Hitchin) pairs $(\mathcal{E}, \theta)$ where:

- $\mathcal{E}$ rank n vector bundle on Γ,
- $\theta : \mathcal{E} \to \mathcal{E}(\Delta)$,

$\mathcal{A} = \bigoplus_{i=1}^{n} H^0(\Gamma, \mathcal{O}_\Gamma(i\Delta))$.

Hitchin fibration: $m : \mathcal{M} \to \mathcal{A}$ with

$$m(\mathcal{E}, \theta) = (-\mathrm{tr}(\theta), \mathrm{tr}(\wedge^2\theta), \ldots, (-1)^n \mathrm{tr}(\wedge^n\theta))$$

SPECTRAL CURVES

$p : \Sigma = \mathbb{V}(\mathcal{O}_\Gamma(-\Delta)) \to \Gamma$ ruled surface,
$u \in H^0(\Sigma, p^*\mathcal{O}_\Gamma(\Delta))$ universal section.

$a \in \mathcal{A} \Rightarrow$ the spectral curve $C_a \subset \Sigma$ with equation:

$$C_a = \{u^n + p^*a_1 \cdot u^{n-1} + \cdots + p^*a_n = 0\},$$

$p : C_a \to \Gamma$ is a finite ramified covering of degree n.

Proposition (Beauville-Narasimhan-Ramanan)

$\forall\ a \in \mathcal{A}$ such that C_a is reduced, $m^{-1}(a) = \mathcal{M}_a$ is canonically isomorphic to $\overline{\mathrm{Pic}}(C_a)$.

ABOUT OUR PROOF

- First of all we work with the Hitchin fibration for an unramified unitary group scheme over Γ.
- Next we use Goresky-Kottwitz-MacPherson approach via equivariant cohomology, but now in family.
- The required purity conjecture follows from Deligne's purity theorem for $Rm_*\mathbb{Q}_\ell$ where $m : \mathcal{M} \to \mathcal{A}$ is the Hitchin fibration.