On optimal control problems with mixed control-state constraints

Arnd Rösch

Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz
Austrian Academy of Sciences

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Outline

1. Motivation
2. Lavrentiev regularization
   - Regularization error
   - Stability results
   - Numerical tests
3. Discretization

Cooperation

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- Svetlana Cherednichenko
- Klaus Krumbiegel

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Contents

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A state constrained optimal control problem

Objective

$$\min J(u) = F(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)}$$

State equation

$$\begin{aligned} Ay &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma \end{aligned}$$

Constraints

$$\begin{array}{ccccll} y_c & \le & y & & & \text{a.e. in } \Omega' \subset \Omega \\ 0 & \le & u & \le & b & \text{a.e. in } \Omega \end{array}$$

Lavrentiev regularization

Assume that $\bar{y} = y_c$ holds on a subdomain. $\rightarrow$ The optimal control is obtained by twice differentiating the data:

$$Ay_c = u.$$

Consequently, we have

- $\rightarrow$ some properties of ill-posed problems
- $\rightarrow$ in particular high condition numbers after discretization

Moreover, the Lagrange multiplier μ associated to the state constraints are only Borel measures (Dirac measures are possible).
One possible way out: $\rightarrow$ Lavrentiev regularization

Lavrentiev regularization

$$\pm\varepsilon u + y \geq y_c$$

Contents

1 Motivation

2 Lavrentiev regularization
- Regularization error
- Stability results
- Numerical tests

3 Discretization

Lavrentiev regularization

We use a Lavrentiev type regularization of the state constraints.

Regularized state constraints

$$\pm\varepsilon u + y \geq y_c$$

Known results: Existence of regular Lagrange multipliers in several cases:

- linear quadratic problems → Tröltzsch (2004),
- semilinear problems → Rösch and Tröltzsch (2005, 2006).

Optimal controls are Lipschitz continuous. Convergence of the optimal controls $\bar{u}_\varepsilon$ can be shown for $\varepsilon \downarrow 0$ (Meyer, Rösch and Tröltzsch (2004)). Next, we will discuss the approach via

$$\varepsilon u + y \geq y_c.$$

Regularity condition - Existence of optimal solutions

We assume the existence of an "inner point":

Regularity condition - existence of an inner point

$$\exists \hat{u} \in L^2(\Omega): \; 0 \le \hat{u} \le b \quad \text{and} \quad \hat{y} \ge y_c + \tau, \quad \tau > 0$$

Existence of optimal solutions

Lemma: The unregularized problem admits a unique solution $\bar{u}$.
Moreover, the regularized problems admit unique solutions $\bar{u}_\varepsilon$.

Remark

An additional smallness condition for ε is needed to ensure this result in the case

$$-\varepsilon u + y \ge y_c.$$

Admissible control sets and optimality conditions

We define the following sets:

Admissible control sets

$$\begin{aligned} U_{ad} &:= \{u \in L^2(\Omega) : 0 \le u \le b,\ y_c \le y, \} \\ U_{ad}^{\varepsilon} &:= \{u \in L^2(\Omega) : 0 \le u \le b,\ y_c \le y + \varepsilon u\} \end{aligned}$$

Here, y denotes always the associated state to u.

Optimality conditions

$$\begin{aligned} (\nu \bar{u} + \bar{p}, u - \bar{u}) &\geq 0 \quad \text{for all } u \in U_{ad} \\ (\nu \bar{u}_\varepsilon + \bar{p}_\varepsilon, u - \bar{u}_\varepsilon) &\geq 0 \quad \text{for all } u \in U_{ad}^{\varepsilon} \end{aligned}$$

where the adjoint states $\bar{p}$ and $\bar{p}_\varepsilon$ are defined via an adjoint equation:

Adjoint equation

$$\begin{aligned} A^* p &= y - y_d \quad \text{in } \Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Construction of test functions

Construction of a test function for the first variational inequality

Lemma: The control $u_\delta := (1-\delta)\bar{u}_\varepsilon + \delta\hat{u}$ is feasible for (P) for $\delta \in [\delta_\varepsilon, 1]$ with

$$\delta_\varepsilon = \frac{\varepsilon b}{\tau + \varepsilon b}.$$

The assertion is easily obtained using the specific properties of the inner point $\hat{u}$.

Construction of a test function for the second variational inequality

Lemma: $\bar{u}$ belongs to U_{ad}^{ε} for arbitrary $\varepsilon > 0$.

The assertion is true because of

$$\varepsilon\bar{u} + \bar{y} \geq \bar{y} \geq y_c.$$

Error estimates

Adding the two variational inequalities with our specific test functions, we obtain

$$\nu\|\bar{u} - \bar{u}_\varepsilon\|^2_{L^2(\Omega)} + \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2(\Omega)} \le (\nu\bar{u} + \bar{p},\ u_\delta - \bar{u}_\varepsilon).$$

Moreover, we have

$$\|u_\delta - \bar{u}_\varepsilon\|_{L^2(\Omega)} = \delta\|\hat{u} - \bar{u}_\varepsilon\|_{L^2(\Omega)} \le \delta b|\Omega|^{\frac{1}{2}}.$$

Setting $\delta = \delta_\varepsilon$ we end up with

Regularization error

$$\nu\|\bar{u}_\varepsilon - \bar{u}\|^2_{L^2(\Omega)} + \|\bar{y}_\varepsilon - \bar{y}\|^2_{L^2(\Omega)} \le c\varepsilon.$$

Stability of regularized solutions

Assumption: The data y_d and y_c are perturbed:

Noisy data

$$\begin{aligned} \|y_d - y_d^\sigma\|_{L^2(\Omega)} &\leq \sigma_d \\ \|y_c - y_c^\sigma\|_{L^\infty(\Omega)} &\leq \sigma_c \end{aligned}$$

We modify the regularity condition

Regularity condition - existence of an inner point

$$\exists \hat{u} \in L^2(\Omega): \; 0 \leq \hat{u} \leq b \quad \text{and} \quad \hat{y} \geq y_c + \tau, \quad \tau - \sigma_c = \tau' > 0.$$

That means that the safety parameter τ is larger than the noise level δ. We denote by $\bar{u}_\varepsilon^\sigma$ the solution of the regularized problem with noisy data. Our goal is to estimate the distance $\|\bar{u}_\varepsilon^\sigma - \bar{u}\|$ to the solution of the unregularized problem with exact data.

Existence of optimal solutions and optimality system

Existence of optimal solutions

Lemma: The unregularized problem admits a unique solution $\bar{u}$.
Moreover, the regularized problems admit unique solutions $\bar{u}^\sigma_\varepsilon$.

The assertion is true, since the modified regularity condition ensures the existence of a feasible point.

Optimality conditions

$$\begin{aligned} (\nu\bar{u} + \bar{p}, u - \bar{u}) &\geq 0 \quad \text{for all } u \in U_{ad} \\ (\nu\bar{u}^\sigma_\varepsilon + \bar{p}^\sigma_\varepsilon, u - \bar{u}^\sigma_\varepsilon) &\geq 0 \quad \text{for all } u \in U^{\varepsilon,\delta}_{ad} \end{aligned}$$

The noisy data influence the adjoint equation:

Adjoint equation for the problem with noisy data

$$\begin{aligned} A^* p^\sigma_\varepsilon &= y^\sigma_\varepsilon - y^\sigma_d && \text{in } \Omega \\ p^\sigma_\varepsilon &= 0 && \text{on } \partial\Omega \end{aligned}$$

Construction of test functions

The noisy data influence the construction of the test functions

Construction of test functions

Lemma: The control $u_\delta^\sigma := (1-\delta)\bar{u} + \delta\hat{u}$ is feasible for (P_ε^σ) for $\delta \in [\delta_\varepsilon^\sigma, 1]$. Moreover, $u_\varrho^\sigma := (1-\varrho)\bar{u}_\varepsilon^\sigma + \varrho\hat{u}$ is feasible for (P) for every ϱ in $[\varrho_\varepsilon^\sigma, 1]$.

The quantities $\delta_\varepsilon^\sigma$, $\varrho_\varepsilon^\sigma$ are given by

$$\delta_\varepsilon^\sigma = \frac{\sigma_C}{\sigma_C + \tau}, \qquad \varrho_\varepsilon^\sigma = \frac{\varepsilon b + \sigma_C}{\varepsilon b + \sigma_C + \tau'}$$

Using these two test functions, we find the estimate

$$\begin{aligned}\nu\|\bar{u} - \bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 + \|\bar{y} - \bar{y}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 \quad \le \quad & (\nu\bar{u} + \bar{p}, u_\varrho^\sigma - \bar{u}_\varepsilon^\sigma) \\ & + (\bar{p}_\varepsilon^\sigma + \nu\bar{u}_\varepsilon^\sigma, u_\delta^\sigma - \bar{u}) \\ & + (y_d - y_d^\sigma, \bar{y} - \bar{y}_\varepsilon^\sigma).\end{aligned}$$

Stability estimate

Estimating all terms, we find the stability estimate

Stability estimate

$$\nu\|\bar{u} - \bar{u}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\bar{y} - \bar{y}_\varepsilon^\sigma\|_{L^2(\Omega)}^2 \leq C_1\varepsilon + C_2\sigma_c + \frac{1}{2}\sigma_d^2.$$

The constants C_1 and C_2 can be expressed as follows

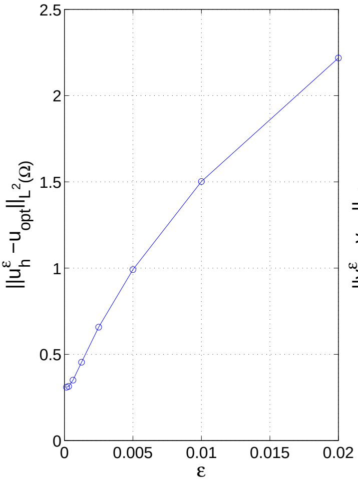
$$\begin{aligned} C_1 &= \frac{b^2}{\tau'}|\Omega|^{\frac{1}{2}} \cdot \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)} \\ C_2 &= \frac{b}{\tau'}|\Omega|^{\frac{1}{2}} \cdot \|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)} + \frac{b}{\tau}|\Omega|^{\frac{1}{2}} \cdot \|\nu\bar{u}_\varepsilon^\sigma + \bar{p}_\varepsilon^\sigma\|_{L^2(\Omega)}. \end{aligned}$$

Using the inner point, it is easy to find a priori bounds for the expressions $\|\nu\bar{u} + \bar{p}\|_{L^2(\Omega)}$ and $\|\nu\bar{u}_\varepsilon^\sigma + \bar{p}_\varepsilon^\sigma\|_{L^2(\Omega)}$

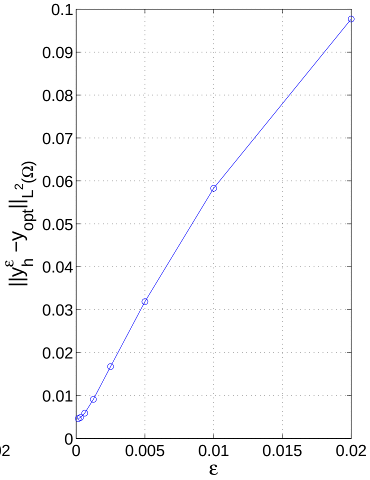
Feasible solution

The control u_ϱ^σ is feasible for (P) and fulfills the same error estimate.

Numerical test - dependence on ε



error in the control



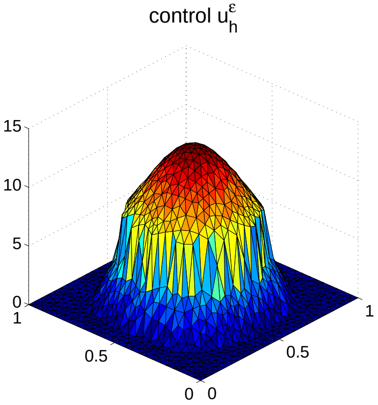
error in the state

Numerical test - dependence on ε

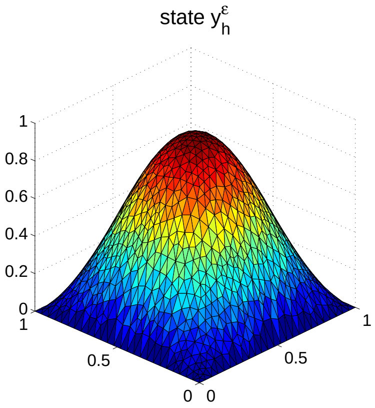
ε	$\|\bar{u} - u_h^\varepsilon\|_{L^2(\Omega)}$	$\frac{\|\bar{u} - u_h^\varepsilon\|_{L^2(\Omega)}}{\sqrt{\varepsilon}}$	$\|\bar{y} - y_h^\varepsilon\|_{L^2(\Omega)}$
$2^1 \cdot 10^{-2}$	$2.2180e+0$	15.684	$9.7706e-2$
$2^0 \cdot 10^{-2}$	$1.5018e+0$	15.018	$5.8289e-2$
$2^{-1} \cdot 10^{-2}$	$9.9099e-1$	14.015	$3.1875e-2$
$2^{-2} \cdot 10^{-2}$	$6.5758e-1$	13.152	$1.6745e-2$
$2^{-3} \cdot 10^{-2}$	$4.5411e-1$	12.844	$9.1220e-3$
$2^{-4} \cdot 10^{-2}$	$3.5025e-1$	14.010	$5.9091e-3$
$2^{-5} \cdot 10^{-2}$	$3.1421e-1$	17.774	$4.8770e-3$

ε-dependency

Numerical Tests - solution for $\varepsilon = 0.005$

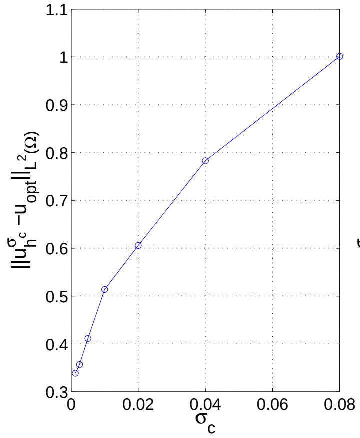


Calculated control

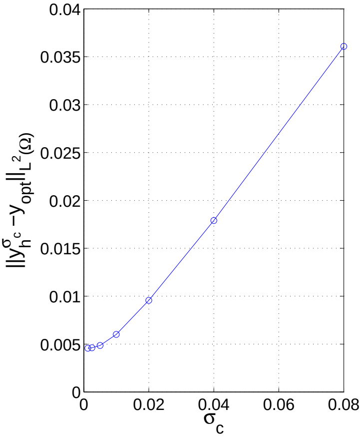


Calculated state

Numerical Tests - dependence on σ_c



error in the control



error in the state

Numerical Tests - dependence on σ_c

σ_c	$\|\bar{u} - u_h^{\varepsilon,\sigma_c}\|_{L^2(\Omega)}$	$\dfrac{\|\bar{u} - u_h^{\varepsilon,\sigma_c}\|_{L^2(\Omega)}}{\sqrt{\sigma_c}}$	$\|\bar{y} - y_h^{\varepsilon,\sigma_c}\|_{L^2(\Omega)}$
$2^3 \cdot 10^{-2}$	$1.0011e+0$	3.5394	$3.6082e-2$
$2^2 \cdot 10^{-2}$	$7.8281e-1$	3.9140	$1.7899e-2$
$2^1 \cdot 10^{-2}$	$6.0590e-1$	4.2844	$9.5663e-3$
$2^0 \cdot 10^{-2}$	$5.1377e-1$	5.1377	$6.0098e-3$
$2^{-1} \cdot 10^{-2}$	$4.1156e-1$	5.8204	$4.8575e-3$
$2^{-2} \cdot 10^{-2}$	$3.5701e-1$	7.1401	$4.6160e-3$

σ_c-dependency

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1 Motivation

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3 Discretization

Finite element discretization

We discuss a standard FE-discretization:

- Piecewise constant or piecewise linear controls
- Piecewise linear finite elements for the state

Remark

In the case of a semidiscretization (only discretization of the PDEs) we can apply the results concerning the stability presented before. The perturbation σ_c repesents now the discretization error

$$\|\bar{u}_\varepsilon - \bar{u}_\varepsilon^h\|_{L^2(\Omega)} \leq c\sqrt{\sigma_c} \leq ch|\ln h|^{1/2}$$

For full discretization we have to modify the estimation strategy.

The fully discretized problem

- We require the existence of an inner point for the undiscretized problem.
- Therefore, it is easy to construct a control u_h^σ which is close to the solution of the fully discretized problem u_h and feasible for the undiscretized problem.
- However, we need also a control u^δ which is close to the solution of the undiscretized problem $\bar{u}$ and feasible for the discretized problem.
- Consequently, we need two ingredients:
  - A piecewise constant (linear) control which is close to $\bar{u}$
  - A piecewise constant (linear) inner point
- This problem is solved using the L^2-projection of both points.

Basic properties and discretization error

FE-error

$$\begin{aligned} \|Sf - S_h f\|_{L^\infty(\Omega')} &\leq ch^2 |\ln h| \|Sf\|_{W^{2,\infty}(\Omega)} \\ \|Sf - S_h f\|_{L^\infty(\Omega')} &\leq ch^2 (\ln h)^2 \|f\|_{L^\infty(\Omega)} \end{aligned}$$

The second inequality is essentially needed because of the piecewise constant controls. Estimating all terms, we end up with:

Discretization error

$$\|\bar{u}_\varepsilon - \bar{u}_\varepsilon^h\|_{L^2(\Omega)} \leq ch|\ln h| + f(\varepsilon) \cdot h^{1/2}$$

Moreover, we have

$$f(\varepsilon) \to 0 \quad \text{for} \quad \varepsilon \to 0.$$

Challenge for the future: Optimal tuning of h and ε

Summary

- Optimal control problems with pointwise state constraints can be regularized with a Lavrentiev type regularization.
- The regularization error can be estimated.
- The solutions are stable with respect to noisy data.
- It is possible to construct feasible approximations.
- Estimates for the discretization error are available.