A K-theoretic Proof of Boutet de Monvel's Index Theorem for Boundary Value Problems

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Coauthors

Relies on joint work with

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1. Melo, Nest, Schrohe. C^*-structure and K-theory of Boutet de Monvel's algebra.
J. Reine Angew. Math. 2003.
2. Melo, Schick, Schrohe. A K-theoretic proof of Boutet de Monvel's index theorem for boundary value problems. math.KT/0403059,
J. Reine Angew. Math. (to appear)

The Index

Fredholm Operators

A linear operator P is a *Fredholm operator* if

$$\dim \ker P \quad \text{and} \quad \operatorname{codim} \operatorname{ran} P \quad \text{are both finite.}$$

In that case

$$\operatorname{Index} P = \dim \ker P - \operatorname{codim} \operatorname{ran} P \in \mathbb{Z}.$$

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Important

Index is stable under small and compact perturbations.

The Index

The Classical Situation: Closed Manifold M

P (pseudo-)differential operator

$$P : C^\infty(M, E) \longrightarrow C^\infty(M, F)$$

acting between sections of vector bundles E, F over M.
Elliptic: Principal symbol $\sigma_P(x, \xi)$ invertible for $(x, \xi) \in T^*M \setminus 0$.

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Central Facts

- Ellipticity implies that P is Fredholm.
- Index depends only on principal symbol (lower order terms = compact perturbations)
- Index depends only on stable homotopy classes of σ_P.

The Index Problem

Gelfand 1960

Compute Index P from σ_P.

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Atiyah und Singer 1963

- Solved the problem.
- Key tools: K-theory and pseudodifferential calculus.

K-theory

Definition

A K-class with compact support over X is a triple (E, F, σ)

- E, F vector bundles over X
- $\sigma : E \to F$ vector bundle map
- σ isomorphism outside compact set.

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Principal Symbol

σ_P 'lives' on $T^*M \setminus 0$. Defines homomorphism $\pi^*E \to \pi^*F$.
Moreover: Isomorphism outside zero section due to ellipticity.
Hence: Defines an element

$$[\sigma_P] \in K_c(T^*M).$$

Index Theorem

Topological Index Map

There exists a homomorphism (‘topological index map’)

$$\chi_t : K_c(T^*M) \to \mathbb{Z}$$

Corollary

Have two ways of associating an integer to an elliptic operator:

- Take the Fredholm index of P
- Take the topological index of $[\sigma_P]$

Index Theorem: Same result

$$\text{Index}\, P = \chi_t([\sigma_P]).$$

Index Theorem

Cohomological Form

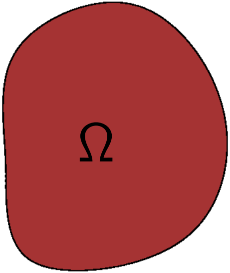
$$\text{Index}\, P = \int \mathrm{ch}\,[\sigma_P] \wedge \mathrm{Td}\,(M)$$

with Chern character of the K-class and the Todd genus of M.

Classical Boundary Value Problems

$\Omega \subseteq \mathbb{R}^n$ smoothly bounded domain.
P differential operator on Ω, f function on Ω,
T trace operator, g function on $\partial\Omega$.
Find u on $\overline{\Omega}$ with

$$Pu = f \text{ in } \Omega \quad \text{and} \quad Tu = g \text{ on } \partial\Omega.$$



Example: Dirichlet Problem

$$\begin{aligned} \Delta u &= f \text{ in } \Omega \\ \gamma_0 u = u|_{\partial\Omega} &= g \text{ auf } \partial\Omega. \end{aligned}$$

Ellipticity

Lopatinskij-Shapiro Condition

The boundary problem $\binom{P}{T}$ is elliptic, if

- P is elliptic and
- for each $(x', \xi') \in T^*(\partial X) \setminus 0$

$$\begin{pmatrix} \sigma_P(x', 0, \xi', D_n) \\ \sigma_T(x', \xi', D_n) \end{pmatrix} : \mathcal{S}(\mathbb{R}_+) \xrightarrow{\cong} \begin{matrix} \mathcal{S}(\mathbb{R}_+) \\ \oplus \\ \mathbb{C} \end{matrix}$$

boundary symbol must be invertible

Here, locally $X = \{x_n \geq 0\}$.

Detour: Solving the Dirichlet Problem

Solving the Dirichlet Problem

Solution is a sum $u = u_1 + u_2$, where u_1 and u_2 solve

$$\begin{aligned} \Delta u_1 &= f \quad \text{and} \quad & \Delta u_2 &= 0 \\ \gamma_0 u_1 &= 0 & \gamma_0 u_2 &= g. \end{aligned}$$

Obtain

u_1 using

Green's function Γ

$u_1 = \Gamma f = \int_\Omega \Gamma(x,y) f(y) dy$

u_2 using

Poisson operator K

$u_2 = Kg = \int_{\partial\Omega} K(x,y) g(y) dy$

Detour: Solving the Dirichlet problem

Green's function Γ is the sum of

- Newton potential (= fundamental solution of Δ) and
- correction term (= smooth in the interior).

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Green's function Γ is the sum of

- Newton potential (= fundamental solution of Δ) P and
- correction term G (= smooth in the interior).

As an operator:

$$\begin{pmatrix} \Delta \\ \gamma_0 \end{pmatrix} : C^\infty(\overline{\Omega}) \longrightarrow \begin{matrix} C^\infty(\overline{\Omega}) \\ \oplus \\ C^\infty(\partial\Omega) \end{matrix}$$

Inverse:

$$\begin{pmatrix} \Delta \\ \gamma_0 \end{pmatrix}^{-1} = \big(\underbrace{P + G}_{\Gamma} \;\; K \big) : \begin{matrix} C^\infty(\overline{\Omega}) \\ \oplus \\ C^\infty(\partial\Omega) \end{matrix} \longrightarrow C^\infty(\overline{\Omega})$$

Boutet de Monvel's Algebra

Goal (Boutet de Monvel 1971)

Construction of an algebra containing

- the classical boundary value problems and
- their (pseudo-)inverses, whenever those exist.

Boutet de Monvel's Algebra

X smooth compact manifold with boundary.
An operator in Boutet de Monvel's algebra is a matrix

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X, E_1) \\ \oplus \\ C^\infty(\partial X, F_1) \end{matrix} \longrightarrow \begin{matrix} C^\infty(X, E_2) \\ \oplus \\ C^\infty(\partial X, F_2) \end{matrix}$$

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- S pdo on ∂X.

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Note:

- Contains classical boundary value problems:
  $F_1 = 0$, $G = 0$, K, S not present.
- Contains their inverses (if they exist):
  $F_2 = 0$, T, S not present.
- Allows composition, if bundles match.
  $\longrightarrow$ Algebra for $E_1 = E_2 = E$, $F_1 = F_2 = F$

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Ellipticity in Boutet de Monvel's Algebra

Two Symbols

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix}$$

- Interior symbol: $\sigma(A) = \sigma_P$ on $T^*X \setminus 0$
- boundary symbol $\gamma(A)$ on $T^*\partial X \setminus 0$

$$\gamma(A) = \begin{pmatrix} p_0(x', 0, \xi', D_n) + g_0(x', \xi', D_n) & k_0(x', \xi', D_n) \\ t_0(x', \xi', D_n) & s_0(x', \xi') \end{pmatrix}$$

Ellipticity = Invertibility of both symbols $\longrightarrow$ Fredholm operator.
Index determined by two symbols.

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- Main Step: An elliptic operator A as above is stably homotopic to an operator of the form

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- Can associate to A a class $[A]$ in $K_c(T^*X^\circ)$ by letting

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- Using the Atiyah-Singer Index Theorem, he obtains

$$\text{Index}\,A = \chi_t([A]).$$

Boutet de Monvel's Index Theorem

Cohomological form (Fedosov 1996)

$$\operatorname{Index} A = \int_{T^*X} \mathrm{ch}(\sigma(A))\mathrm{Td}(X) + \int_{T^*\partial X} \mathrm{ch}'(\gamma(A))\mathrm{Td}(\partial X).$$

K-theory

Reductions: – X connected, $\partial X \neq \emptyset$
– Endomorphisms of order and type zero.

Definition

$\mathfrak{A} = C^*-$closure of operators of order and class 0
$\mathfrak{K} =$ ideal of compact operators.

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Theorem

Find natural short exact sequences

$$0 \to K_i(C(X)) \longrightarrow K_i(\mathfrak{A}/\mathfrak{K}) \xrightarrow{p} K_{1-i}(C_0(T^*X^\circ)) \to 0,$$

$i = 0, 1$. The sequences split (though not naturally), and

$$K_i(\mathfrak{A}/\mathfrak{K}) = K_i(C(X)) \oplus K_{1-i}(C_0(T^*X^\circ)).$$

K-theoretic Version of the Index Theorem

Theorem

The map p in the short exact sequence

$$0 \to K_1(C(X)) \longrightarrow K_1(\mathfrak{A}/\mathfrak{K}) \xrightarrow{p} K_0(C_0(T^*X^\circ)) \to 0,$$

is Boutet de Monvel's map. With the topological index map χ_t

$$\operatorname{Index} A = \chi_t(p(A)).$$

Also Fedosov's cohomological formula follows.

K-theoretic Version of the Index Theorem

Comparison with Boutet de Monvel

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- His constructions are very geometric. Uses classical K-theory
- Our proof relies on
  – knowledge of algebra structure of Boutet de Monvel's algebra
  – K-theory of C^*-algebras (not yet developed in 1971!)
  – standard constructions in K-theory.

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- Boutet de Monvel's proof is ingeneous, but hard to understand.
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- Our proof relies on
  – knowledge of algebra structure of Boutet de Monvel's algebra
  – K-theory of C^*-algebras (not yet developed in 1971!)
  – standard constructions in K-theory.
- Much simpler, but sometimes less explicit.

Idea of the Proof

Understand Boundary Symbol

$\gamma : \mathfrak{A} \to C(S^*\partial X, \mathfrak{W})$ $\mathfrak{W}$ Wiener-Hopf operators ($\approx$ Toeplitz).

$$\gamma(A) = \begin{pmatrix} p_0(x', 0, \xi', D_n) + g_0(x', \xi', D_n) & k_0(x', \xi', D_n) \\ t_0(x', \xi', D_n) & s_0(x', \xi') \end{pmatrix}$$

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- $\ker \gamma = \{A : p^0 = 0 \text{ at } \partial X, G, T, K, S \text{ lower order.}\}$. Contains compact operators.

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- $\operatorname{ran} \gamma = C(\partial X) \oplus C(S^*\partial X, \mathfrak{W}_0)$.
  $\mathfrak{W}_0$: Ideal of operators, for which symbol vanishes at ∞.

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- $\operatorname{ran} \gamma = C(\partial X) \oplus C(S^*\partial X, \mathfrak{W}_0)$. $\mathfrak{W}_0$: Ideal of operators, for which symbol vanishes at ∞.
- $K_i(\mathfrak{W}_0) = 0 \Rightarrow K_i(C(S^*\partial X, \mathfrak{W}_0)) = 0$.

Idea of the Proof

Understand Short Exact Sequence

$$0 \to \ker\gamma/\mathfrak{K} \to \mathfrak{A}/\mathfrak{K} \to \operatorname{ran}\gamma = \mathfrak{A}/\ker\gamma \to 0.$$

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- $K_i(\operatorname{ran}\gamma) \cong K_i\big(C(\partial X)\big) \oplus \underbrace{K_i(C(S^*\partial X, \mathfrak{W}_0))}_{=0} = K_i(C(\partial X))$

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Understand Short Exact Sequence

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- $K_i(\operatorname{ran}\gamma) \cong K_i\big(C(\partial X)\big) \oplus \underbrace{K_i(C(S^*\partial X, \mathfrak{W}_0))}_{=0} = K_i(C(\partial X))$
- Isomorphism implemented by

$$C(\partial X) \ni g \stackrel{b}{\mapsto} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{A}/\mathfrak{K},$$

where f is a function in $C(X)$ with $f = g$ on ∂X, considered as multiplication operator.

Idea of the Proof

The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\gamma/\mathfrak{K} & \longrightarrow & \mathfrak{A}/\mathfrak{K} & \xrightarrow{\pi} & \operatorname{ran}\gamma & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle m_0} & & \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle b} & & \\ 0 & \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \xrightarrow{r} & C(\partial X) & \longrightarrow & 0 \end{array} .$$

induces canonically a grid:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \stackrel{r}{\longrightarrow} & C(\partial X) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Cm_0 & \longrightarrow & Cm & \longrightarrow & Cb & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
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 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

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0 & \longrightarrow & Cm_0 & \longrightarrow & Cm & \longrightarrow & Cb & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S(\ker\gamma/\mathfrak{K}) & \longrightarrow & S(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{S\pi} & S(\operatorname{ran}\gamma) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

- b isomorphism $\Rightarrow$ (right vertical) $K_i(Cb) = 0$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \stackrel{r}{\longrightarrow} & C(\partial X) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Cm_0 & \longrightarrow & Cm & \longrightarrow & Cb & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S(\ker\gamma/\mathfrak{K}) & \longrightarrow & S(\mathfrak{A}/\mathfrak{K}) & \stackrel{S\pi}{\longrightarrow} & S(\operatorname{ran}\gamma) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

- b isomorphism $\Rightarrow$ (right vertical) $K_i(Cb) = 0$
- $\Rightarrow$ (middle horizontal) $K_i(Cm) \cong K_i(Cm_0)$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_0(X^\circ) & \longrightarrow & C(X) & \xrightarrow{r} & C(\partial X) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & Cm_0 & \longrightarrow & Cm & \longrightarrow & Cb & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S(\ker\gamma/\mathfrak{K}) & \longrightarrow & S(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{S\pi} & S(\operatorname{ran}\gamma) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

- b isomorphism $\Rightarrow$ (right vertical) $K_i(Cb) = 0$
- $\Rightarrow$ (middle horizontal) $K_i(Cm) \cong K_i(Cm_0)$.
- Study long exact sequence for left verticals.

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
& \uparrow & & i\uparrow & & \phi\uparrow\cong & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
& \uparrow & & i\uparrow & & \phi\uparrow\cong & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}$$

- Principal symbol furnishes Iso $\ker\gamma/\mathfrak{K} \cong C_0(S^*X^\circ)$.

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
& \uparrow & & \uparrow{\scriptstyle i} & & {\scriptstyle \phi}\uparrow{\scriptstyle \cong} & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}$$

- Principal symbol furnishes Iso $\ker\gamma/\mathfrak{K} \cong C_0(S^*X^\circ)$.
- X connected, $\partial X \neq \emptyset \Rightarrow S^*X^\circ$ has a section.

$$
\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
 & \uparrow & & i\uparrow & & \phi\uparrow\cong & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}
$$

- Principal symbol furnishes Iso $\ker\gamma/\mathfrak{K} \cong C_0(S^*X^\circ)$.
- X connected, $\partial X \neq \emptyset \Rightarrow S^*X^\circ$ has a section.
- Yields $s : C_0(S^*X^\circ) \to C(X^\circ)$ left inverse to m_{0*}.

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
& \uparrow & & \uparrow i & & \phi \uparrow \cong & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}$$

- Principal symbol furnishes Iso $\ker\gamma/\mathfrak{K} \cong C_0(S^*X^\circ)$.
- X connected, $\partial X \neq \emptyset \Rightarrow S^*X^\circ$ has a section.
- Yields $s : C_0(S^*X^\circ) \to C(X^\circ)$ left inverse to m_{0*}.
- $\Rightarrow$ Sequence splits at m_{0*} (not naturally).

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
 & \uparrow & & i\uparrow & & \phi\uparrow\cong & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}$$

- Principal symbol furnishes Iso $\ker\gamma/\mathfrak{K} \cong C_0(S^*X^\circ)$.
- X connected, $\partial X \neq \emptyset \Rightarrow S^*X^\circ$ has a section.
- Yields $s : C_0(S^*X^\circ) \to C(X^\circ)$ left inverse to m_{0*}.
- $\Rightarrow$ Sequence splits at m_{0*} (not naturally).
- $\Rightarrow$ splits at $\alpha \Rightarrow$ splits at β.

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
 & \uparrow & & \uparrow i & & \phi \uparrow \cong & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}$$

- Principal symbol furnishes Iso $\ker\gamma/\mathfrak{K} \cong C_0(S^*X^\circ)$.
- X connected, $\partial X \neq \emptyset \Rightarrow S^*X^\circ$ has a section.
- Yields $s : C_0(S^*X^\circ) \to C(X^\circ)$ left inverse to m_{0*}.
- $\Rightarrow$ Sequence splits at m_{0*} (not naturally).
- $\Rightarrow$ splits at $\alpha \Rightarrow$ splits at β.
- $\Rightarrow$ long exact sequence furnishes short exact sequences

$$0 \to K_i(C(X)) \xrightarrow{m_*} K_i(\mathfrak{A}/\mathfrak{K}) \xrightarrow{\beta} K_{1-i}(Cm) \to 0,$$

$$\begin{array}{ccccccccc}
\longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{\beta} & K_{1-i}(Cm) & \longrightarrow \\
& \uparrow & & i \uparrow & & \phi \uparrow \cong & \\
\longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\ker\gamma/\mathfrak{K}) & \xrightarrow{\alpha} & K_{1-i}(Cm_0) & \longrightarrow
\end{array}$$

- Principal symbol furnishes Iso $\ker\gamma/\mathfrak{K} \cong C_0(S^*X^\circ)$.
- X connected, $\partial X \neq \emptyset \Rightarrow S^*X^\circ$ has a section.
- Yields $s : C_0(S^*X^\circ) \to C(X^\circ)$ left inverse to m_{0*}.
- $\Rightarrow$ Sequence splits at m_{0*} (not naturally).
- $\Rightarrow$ splits at $\alpha \Rightarrow$ splits at β.
- $\Rightarrow$ long exact sequence furnishes short exact sequences

$$0 \to K_i(C(X)) \xrightarrow{m_*} K_i(\mathfrak{A}/\mathfrak{K}) \xrightarrow{\beta} K_{1-i}(Cm) \to 0,$$

- Now identify $K_{1-i}(Cm) \cong K_{1-i}(C_0(T^*X^\circ))$.

Index Theory

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_i(C(X)) & \xrightarrow{m_*} & K_i(\mathfrak{A}/\mathfrak{K}) & \xrightarrow{p} & K_{1-i}(C_0(T^*X^\circ)) & \longrightarrow & 0 \\
 & & \uparrow & & \iota\uparrow & & \phi\uparrow\cong & & \\
0 & \longrightarrow & K_i(C_0(X^\circ)) & \xrightarrow{m_{0*}} & K_i(\mathfrak{I}/\mathfrak{K}) & \longrightarrow & K_{1-i}(Cm_0) & \longrightarrow & 0
\end{array}$$

Show $\operatorname{Index} A = \chi \circ p(A)$ on the ranges of $\iota_* : K_i(\ker \gamma/\mathfrak{K}) \to K_i(\mathfrak{A}/\mathfrak{K})$ and $m_* : K_i(C(X)) \to K_i(\mathfrak{A}/\mathfrak{K})$.

- On $\operatorname{ran} m_*$, both are zero:
  $\operatorname{ran} m_*$: Equivalence classes of invertible multiplication operators. (Index =0).

  On the other hand, exactness of the sequence implies that $\operatorname{ran} m_* \to 0$.
- On $\operatorname{ran} \iota_*$ use Atiyah-Singer Theorem.